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MATHEMATICS

On Cardinal Numbers Related with Locally Compact Groups

by

A. HULANICKI

Presented by A. MOSTOWSKI on May 27, 1957

In this paper we are concerned with the powers of locally compact topological groups. We shall prove that the power of a group belonging to a "very large" class of locally compact groups, including the class of compact groups and that of locally compact connected groups, is of the form $2^{\aleph}$.

The method of proof of the fundamental lemma used here is similar to that used in [1].

1. Definitions. Let G be any locally compact group. We denote by $\Theta(G)$ the minimal power of a family of open sets having in intersection only the unity of G.

By $\tau(X)$ we denote the minimal power of the family of open sets such that each open set of the space X is a sum of some sets of the family.

We say that the space X is $\aleph$-compact if $X = \bigcup_{t \in T} X_t$ is compact and $\overline{\overline{T}} = \aleph$. (If X is $\aleph^*$-compact, then X is $\aleph$-compact with $\aleph \leqslant \aleph^*$).

2. FUNDAMENTAL LEMMA. *If G is a locally compact group, then* $\overline{\overline{G}} \geqslant 2^{\Theta(G)}$.

Proof. We are going to define a transfinite sequence

$$(*) \qquad R^0, \dots, R^a, \dots$$

the terms of which are families R^a of compact subsets of the group G. The family R^a includes sets $A^a_{i_1}, \dots, i_\lambda, \dots$ with a (possibly infinite) sequence of indices $i_1, \dots, i_\lambda, \dots$ of the type a, each of indices taking values 0 or 1.

The sequence (*) is defined by transfinite induction simultaneously with an auxiliary family of open sets $V_{i_1,\dots,i_\lambda,\dots}$.

Let $\mathfrak{V}$ be any open set with compact closure contained in G, and V any other open set such that $\overline{V} \subset \mathfrak{V}$.

Put $R^0 = (A^0) = (\overline{V})$.

Let us assume that we have defined all R^β, where $\beta < \alpha$.

Consider two cases: (i) $\alpha = \alpha' + 1$, (ii) α is a limit-number.

(i) If any of $A^{\alpha'}_{i_1,\ldots,i_\lambda,\ldots}$ belonging to $R^{\alpha'}$ is void or consists of one element, then let $R^{\alpha'}$ be the last element of our sequence. In the other case take $x_1, x_2 \in A^{\alpha'}_{i_1,\ldots,i_\lambda,\ldots}$ $(x_1 \neq x_2)$ and two neighbourhoods $V_{i_1,\ldots,i_\lambda,\ldots,0} \subset \mathfrak{B} \subset G$ and $V_{i_1,\ldots,i_\lambda,\ldots,1} \subset \mathfrak{B} \subset G$ such that $x_1 \in V_{i_1,\ldots,i_\lambda,\ldots,0}$ and $x_2 \in V_{i_1,\ldots,i_\lambda,\ldots,1}$ and $\overline{V}_{i_1,\ldots,i_\lambda,\ldots,0} \cap \overline{V}_{i_1,\ldots,i_\lambda,\ldots,1} = 0$.

Let us form the family $\mathfrak{W}^{i_1,\ldots,i_\lambda,\ldots}_{\alpha'}$ of all finite intersections of sets $V_{i_1,\ldots,i_\mu}$, where $i_1, \ldots, i_\mu$ are all segments of the sequence $i_1, \ldots, i_\lambda, \ldots$, possessing a last element.

Put

$$A^\alpha_{i_1,\ldots,i_\lambda,\ldots,0} = \bigcap_{W \in \mathfrak{W}^{i_1,\ldots,i_\lambda,\ldots}_{\alpha'}} \overline{W \cap V_{i_1,\ldots,i_\lambda,\ldots,0}}$$

$$A^\alpha_{i_1,\ldots,i_\lambda,\ldots,1} = \bigcap_{W \in \mathfrak{W}^{i_1,\ldots,i_\lambda,\ldots}_{\alpha'}} \overline{W \cap V_{i_1,\ldots,i_\lambda,\ldots,1}}$$

(ii) If α is a limit-number, then

$$A^\alpha_{i_1,\ldots,i_\lambda,\ldots} = \bigcap_{W \in \mathfrak{W}^{i_1,\ldots,i_\lambda,\ldots}_{\alpha}} \overline{W}$$

Now we list some properties of the sequences (*).

Property 1. For all α $A^\alpha_{i_1,\ldots,i_\lambda,\ldots} = \bigcap_{W \in \mathfrak{W}^{i_1,\ldots,i_\lambda,\ldots}_{\alpha}} \overline{W}$.

Property 2. Each family R^α consists only of non-void sets.

Indeed: let first $\alpha = \alpha' + 1$. Take any set $A^\alpha_{i_1,\ldots,i_\lambda,\ldots,0} \in R^\alpha$.

According to the definition, at least two points x_1, x_2 belong to $A^{\alpha'}_{i_1,\ldots,i_\lambda,\ldots} = \bigcap_{W \in \mathfrak{W}^{i_1,\ldots,i_\lambda,\ldots}_{\alpha'}} \overline{W}$ and $x_1 \in V_{i_1,\ldots,i_\lambda,\ldots,0}$.

Hence,

$$x_1 \in \bigcap_{W \in \mathfrak{W}^{i_1,\ldots,i_\lambda,\ldots}_{\alpha'}} \overline{W} \cap V_{i_1,\ldots,i_\lambda,\ldots,0} \subset \bigcap_{W \in \mathfrak{W}^{i_1,\ldots,i_\lambda,\ldots}_{\alpha'}} \overline{\overline{W} \cap V_{i_1,\ldots,i_\lambda,\ldots,0}}.$$

Since $V_{i_1,\ldots,i_\lambda,\ldots,0}$ is an open set $\overline{\overline{W} \cap V_{i_1,\ldots,i_\lambda,\ldots,0}} = \overline{W \cap V_{i_1,\ldots,i_\lambda,\ldots,0}}$.

Hence,

$$x_1 \in \bigcap_{W \in \mathfrak{W}^{i_1,\ldots,i_\lambda,\ldots}_{\alpha'}} \overline{W \cap V_{i_1,\ldots,i_\lambda,\ldots,0}} = A^\alpha_{i_1,\ldots,i_\lambda,\ldots,0} \neq 0.$$

If α is a limit-number, then in order to get

$$\bigcap_{W \in \mathfrak{W}_{\alpha}^{i_1,\ldots,i_\lambda,\ldots}} \overline{W} \neq 0$$

it suffices to show that each set W is non-void and use the compactness of $\overline{\mathfrak{V}}$. Suppose that a set W is void. Write $W = V_{i_1,\ldots,i_{\lambda_1}} \cap \ldots \cap V_{1,\ldots,i_{\lambda_n}}$, where $i_1, \ldots, i_{\lambda_s}$ $(s \leqslant n)$ are segments of $i_1, \ldots, i_{\lambda_s}$. Then by Property 1 the set $A^{\beta}_{i_1,\ldots,i_{\lambda_n}}$ (where β is the ordinal of the sequence $i_1, \ldots, i_{\lambda_n}$), would be void, which is impossible.

Property 3. For any $0-1$ sequence $i_1, \ldots, i_\lambda, \ldots$ of α type there is a corresponding set $A^{\alpha}_{i_1,\ldots,i_\lambda,\ldots}$.

Property 4. For each α all sets of the family R^{α} are disjoint.

Property 5. By Properties 2, 3 and 4, if the family R^{α} appears in the sequence (*), then $\overline{\overline{G}} \geqslant 2^{\overline{\alpha}}$.

It follows from (i) that the sequence (*) has a last element, R^{Θ}, say. By Property 2 there is a set of this family for which $A^{\Theta}_{i_1,\ldots,i_\lambda,\ldots} = (x)$. By Property 5 $\overline{\overline{G}} \geqslant 2^{\overline{\Theta}}$. Hence, it suffices to show that $\Theta(G) \leqslant \overline{\Theta}$ or, which is the same, that there exists a family of power $\overline{\Theta}$, consisting of open sets having in intersection only the unity of G.

By Property 1 we have

$$(x) = \bigcap_{W \in \mathfrak{W}_{\Theta}^{i_1,\ldots,i_\lambda,\ldots}} \overline{W}. \tag{**}$$

We are going to prove that the open sets $x^{-1}WW^{-1}x$; $W \in \mathfrak{W}_{\Theta}^{i_1,\ldots,i_\lambda,\ldots}$ have for intersection the unity of the group G. It is sufficient to prove that for any neighbourhood U of the unity e there exists such $W_0 \in \mathfrak{W}_{\Theta}^{i_1,\ldots,i_\lambda,\ldots}$ that $x^{-1}W_0 \subset U$. By (**) we have $\bigcap_{W \in \mathfrak{W}_{\Theta}^{i_1,\ldots,i_\lambda,\ldots}} x^{-1}\overline{W} = (e)$, and hence $x^{-1}\overline{\mathfrak{V}} \subset G \subset U \cup \bigcap_{W \in \mathfrak{W}_{\Theta}^{i_1,\ldots,i_\lambda,\ldots}} (x^{-1}\overline{W})$. In view of compactness of $\overline{\mathfrak{V}}$, $x^{-1}\overline{\mathfrak{V}} \subset U \cup \bigcup_{i=1}^{n} (x^{-1}\overline{W}_i)'$; hence, in presence of $W_i \subset \mathfrak{V}, x^{-1}W_1 \cap \ldots x^{-1}W_n \subset U$. But $W_0 = W_1 \cap \ldots \cap W_n$ is contained in $\mathfrak{W}_{\Theta}^{i_1,\ldots,i_\lambda,\ldots}$ and $W_0 \subset xU$, i. e. $x^{-1}W_0 \subset U$, which completes the proof of the lemma.

3. Theorem. *If G is $2^{\Theta(G)}$-compact, then $\overline{\overline{G}} = 2^{\Theta(G)}$.*

Indeed: By lemma, we have $\overline{\overline{G}} \geqslant 2^{\Theta(G)}$. Let $G = \bigcup_{t \in T} X_t$, where X_t are compact and $\overline{\overline{T}} = 2^{\Theta(G)}$. Evidently it suffices to show that $\overline{\overline{X}}_t \leqslant 2^{\Theta(G)}$. Since the sets X_t are compact $\Theta(G) \geqslant \tau(X_t)$. It is a well known fact that for any T_1-topological space X, $\overline{\overline{X}} \leqslant 2^{\tau(X)}$.

The conditions of the theorem are satisfied, e. g. if $\tau(G) \leqslant 2^{\Theta(G)}$, (for G is always $\tau(G)$-compact). The last inequality holds, e. g. in the case when G is connected, and more generally, when it can be generated by a compact neighbourhood of the unity.

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MATHEMATICS

Algebraic Structure of Compact Abelian Groups

by

A. HULANICKI

Presented by A. MOSTOWSKI on December 14, 1957

In his book [3], Kaplansky formulated the problem concerning the algebraic characterization of the class of Abelian groups which admit compact topologies. He remarked that the problem of characterization of those Abelian groups becomes reduced to characterization of two classes of Abelian groups:

1. The class of Abelian divisible groups which admit compact topologies.

2. The class of Abelian reduced groups which admit compact topologies.

The characterization of the first of these classes was given by the author in [2].

Here the algebraic characterization of the second class is given. We shall show that the class of Abelian reduced groups which admit compact topologies is the same as the class of all complete direct sums of finite cyclic group and groups of p-adic integers.

The proof makes the utmost use of the duality theory as presented in Pontrjagin [8] and some results of Łoś [7] and Kulikoff [4], [5].

1. Terms, notations, lemmas. All groups are additively written Abelian groups. We denote: by C_{p^n} the cyclic group of order p^n, by C_{p^∞}, the Prüfer's quasi-cyclic group and by J_p the group of p-adic integers. We denote by $\sum\limits_{t\in T} G_t$ the direct sum of groups G_t, and by $\sum\limits_{t\in T}^{*} G_t$ the complete direct sum of groups G_t.

The characters group (topologized in the known manner [8], p. 242) of a topological group G is denoted by $\hat{G}$.

Let us recall some facts concerning Pontrjagin's duality theory:

1.1. If a topological group G is discrete, then $\hat{G}$ is compact, and *vicé versa* if G is compact, then $\hat{G}$ is discrete. ([9], p. 242);

1.2. If a topological group G is locally compact, then $G = \hat{\hat{G}}$ ([8], p. 256);

1.3. The characters group of a compact reduced group is a torsion group ([3], p. 55);

1.4. If G is a locally compact group and H its closed subgroup, then: $\hat{H} \simeq \hat{G}/N$, where N is closed subgroup of $\hat{G}$ and constitutes the annihilator of the subgroup H, i. e. it consists of all characters χ for which $\chi(H) = 0$. We have $N \simeq \widehat{(G/H)}$ ([8], p. 257);

1.5. If G_t are discrete groups, then $\widehat{\sum_{t\in T} G_t} \simeq \sum_{t\in T}^{*} \hat{G}_t$ ([8], p. 259);

1.6. If G is discrete or compact and H its closed pure subgroup, then the annihilator of the subgroup H is a pure subgroup of the group $\hat{G}$ (see [1], [6]);

1.7. The characters group of the group C_{p^n} is C_{p^n} and of the group C_{p^∞} is J_p ([8], pp. 251, 246).

Theorem 1 (Łoś). *If a pure subgroup of any group G has compact topology, then it is a direct summand of the group G* (see [7]).

In his paper [4] Kulikoff gave a notion of a basic subgroup of a primary group:

The basic subgroup of a primary group G is a pure subgroup B such that B is a direct sum of cyclic groups and such that G/B is divisible.

Theorem 2 (Kulikoff). *Any primary group contains a basic subgroup* (see [4] and [5]).

2. Theorem. *Let G be a compact reduced group, then G is algebraically a complete direct sum of finite cyclic groups and groups of p-adic integers.*

Proof: Let G be a compact reduced group. By 1.3 its characters group $\hat{G}$ is a torsion group and hence is a direct sum of primary groups $\hat{G}_p$ which by 1.5 and 1.2 implies that

$$(*) \qquad G = \sum_p^{*} G_p .$$

By Theorem 2, $\hat{G}_p$ contains a basic subgroup $B = \sum_{t\in T_p} C_{p^{n_t}}$. Since B is pure, its annihilator M is, by 1.4, isomorphic to $\widehat{(\hat{G}_p/B)}$ and, by 1.6, M is a pure subgroup of the group G_p and hence, by Theorem 1, it is a direct summand of G_p

$$(**) \qquad G_p = M + N$$

and, by 1.4, 1.5, $N \simeq G_p/M \simeq \hat{B} \simeq \sum_{t\in T_p} C_{p^{n_t}}$. Since B is a basic subgroup of the group $\hat{G}_p$, the group $\hat{G}_p/B$ is a torsion divisible primary group and hence it is of the form $\hat{G}_p/B \simeq \sum_{s\in S_p} C_{p^\infty}^{(s)}$. Thus the group M has the

form $(\hat{G}_p/B) = \sum_{s\in S_p} J_p^{(s)}$. Hence by (*) and (**), we get

$$G \simeq \sum_p^* \sum_{t\in T_p}^* C_{p^{n_t}} + \sum_p^* \sum_{s\in S_p}^* J_p^{(s)}.$$

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MATHEMATICS

On the Linear Hypothesis in the Theory of Normal Regression

by

W. OKTABA

Presented by O. LANGE on November 12, 1957

1. The assumptions connected with the problem of testing the linear hypothesis in the theory of normal regression are as follows:

Uncorrelated random variables

$$y_i = \mu_i + e_i \quad (i = 1, 2, \ldots, n) \tag{1}$$

with expected values

$$\mu_i = E(y_i) = x_{i1}\beta_1 + x_{i2}\beta_2 + \ldots + x_{ip}\beta_p \tag{2}$$

depending on p ($p < n$) parameters $\beta_1, \beta_2, \ldots, \beta_p$ are normally distributed with the common variance σ^2. The symbols e_i denote residuals, and the symbols x_{ij} ($i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, p$) — elements of a given matrix X with n rows and p columns, marked $X = \underset{np}{X}$. The g independent linear restrictions denoted by the matrix relation

$$G\beta = \eta\,, \tag{3}$$

where $G = \underset{gp}{G}$ is a given matrix and η is a given column vector with components $\eta_1, \eta_2, \ldots, \eta_g$, are imposed on the unknown parameters, $\beta_1, \beta_2, \ldots, \beta_p$ being the components of column vector β.

The problem consists in testing the linear hypothesis

$$H\beta = \nu\,, \tag{4}$$

which says that among given and independent parameter functions $H\beta$, exactly h, linearly independent of (3) (where matrix $H = \underset{hp}{H}$, $h + g < p$, $h > 0$, $g \geqslant 0$), has certain values $\nu_1, \nu_2, \ldots, \nu_h$ that are components of the vector ν.

This problem, with the omission of restrictions (3) and on the assumption that rank r of matrix X is equal to p, was solved by S. Koło-

dziejczyk [1]. The general solution for $r \leqslant p$ (under condition (3)), due to C. R. Rao [2], [3] and [4], may be expressed by the following known theorem:

Under assumptions (1), (2), (3) and the further assumption that the linear hypothesis (4) is true, the random variable

$$\frac{Q_r - Q_a}{f_1} : \frac{Q_a}{f_2} \tag{5}$$

is distributed as F with appropriate numbers of degrees of freedom f_1 and f_2, where $Q_a = \text{Min}\ (\sum^{n} e^2)$ and $Q_r = \text{Min}\ \sum^{n} e^2$; minimalization with respect to $\beta_1, \beta_2, \ldots, \beta_p$ being performed in Q_a without restrictions (4) and in Q_r with these restrictions. When $r = p$, expression (5) assumes the form

$$F = \frac{Q_r - Q_a}{h} : \frac{Q_a}{n-p+g} \tag{6}$$

with h and $n-p+g$ degrees of freedom.

It must be noted that the majority of multiple regression models with matrix X of rank $r < p$ may be transformed by means of reparametrization into models, where the number of independent parameters is equal to the rank of the matrix whose elements are the coefficients of these parameters.

The test of significance F testing the linear hypothesis (4) under assumptions (1), (2), (3) and $r = p$ is based on relation (6), which serves as a starting point.

2. Our aim is to eliminate the minimalization marked in (6) and to present in an explicit form the expressions for the random variables, to be deduced from (6) for different types of general multiple regression models under different variants of the linear hypothesis. The expressions thus obtained are given in Theorems 1, 2 and 3. To find these expressions we had to present the following known theorem [5] in the matrix form which we obtained by applying formula (6).

Let us consider assumptions (1) and (2) or equivalent (respectively) matrix relations

$$y = X\beta + e = X_1\gamma + X_2\delta + e \tag{7}$$

and

$$E(y) = X\beta, \tag{8}$$

where $\beta = \begin{bmatrix} \gamma \\ \delta \end{bmatrix}$, y being the column vector with components $y_1, y_2, \ldots, y_n$, and $\underset{np}{X} = \left[\underset{nq}{X_1} \ \underset{n,p-q}{X_2}\right]$; X_1 and X_2 are submatrices of matrix X, and γ and δ are vectors with the components $\beta_1, \beta_2, \ldots, \beta_q$ and $\beta_{q+1}, \beta_{q+2}, \ldots, \beta_p$,

respectively ($p > q > 0$). Then, under the assumption that the hypothesis $\gamma = \gamma_0$ is true, the random variable

$$(9) \qquad \frac{(\hat{\gamma} - \gamma_0)^* (S^{11})^{-1} (\hat{\gamma} - \gamma_0)}{q} : \frac{(y - X\hat{\beta})^* (y - X\hat{\beta})}{n-p}$$

is distributed as F with q and $n-p$ degrees of freedom, where $\hat{\beta} = \begin{bmatrix} \hat{\gamma} \\ \hat{\delta} \end{bmatrix} = S^{-1}X^*y$ and $S^{11} = \underset{qq}{S^{11}}$ is the submatrix of $S^{-1} = \begin{bmatrix} S^{11} & S^{12} \\ S^{21} & S^{22} \end{bmatrix}$, where $S = \underset{pp}{S} = X^*X$ (the asterisk denoting the transpose of the matrix).

In deducing (9) we made use of (6), of normal equations and of some matrix relations.

THEOREM 1. *Under assumptions* (1) *and* (2), *and supposing hypothesis* $M\beta = \varphi_0$ *to be true, where* $M = \underset{qp}{M}$ *is of rank* q *and* $\varphi_0 = \underset{q1}{\varphi_0}$ *is a specified column vector, the random variable*

$$(10) \qquad \frac{(M\hat{\beta} - \varphi_0)^* (MS^{-1}M^*)^{-1} (M\hat{\beta} - \varphi_0)}{q} : \frac{(y - X\hat{\beta})^* (y - X\hat{\beta})}{n-p}$$

is distributed as F *with* q *and* $n-p$ *degrees of freedom.*

To prove this theorem, we transformed the multiple regression model into model (7) with hypothesis $\gamma = \gamma_0$, and used formula (9).

THEOREM 2. *Under assumptions* (1), (2), (3) *and under the further assumption that linear hypothesis* (4) *is true, the random variable*

$$(11) \qquad \frac{(T\hat{\beta} - \tau)^* (TS^{-1}T^*)^{-1} (T\hat{\beta} - \tau) - (G\hat{\beta} - \eta)^* (GS^{-1}G^*)^{-1} (G\hat{\beta} - \eta)}{h} :$$
$$: \frac{(y - X\hat{\beta})^* (y - X\hat{\beta}) + (G\hat{\beta} - \eta)^* (GS^{-1}G^*)^{-1} (G\hat{\beta} - \eta)}{n-p+g}$$

is distributed as F *with* h *and* $n-p+g$ *degrees of freedom, where* $T = \begin{bmatrix} G \\ H \end{bmatrix}$ *and* $\tau = \begin{bmatrix} \eta \\ \nu \end{bmatrix}$.

To prove this theorem, we expressed Q_a and Q_r from (6) in matrix form by using some relations obtained when proving Theorem 1.

THEOREM 3. *Under assumptions* (1) *and* (2) *and under the further assumption that hypothesis* $\beta = U\alpha$ *is true (where matrix* $U = \underset{p,p-q}{U}$ *is of rank* $p-q$ *and* $\alpha^* = [\alpha_{q+1}, \alpha_{q+2}, \ldots, \alpha_p]$, $q > 0$), *the random variable*

$$(12) \qquad \frac{(S\hat{\beta})^* [S^{-1} - U(U^*SU)^{-1}U^*] S\hat{\beta}}{q} : \frac{(y - X\hat{\beta})^* (y - X\hat{\beta})}{n-p}$$

is distributed as F *with* q *and* $n-p$ *degrees of freedom.*

The proof is similar to the proof of Theorem 1.

3. Theorems 1, 2 and 3 may be used for testing the corresponding null hypothesis. To obtain the test of significance, it is sufficient to perform but a few simple matrix operations. Thus, for the model of the one-way classification, we obtained from (11) the expression

$$\frac{(n-p+1)[a^2b^2-2(c-c^0)abd+(c-c^0)^2d^2]}{\left(b\sum_{i=1}^{p}\frac{t_i^2}{n_i}-d^2\right)\left[b\sum_{i=1}^{p}\sum_{j=1}^{n_i}(y_{ij}-\bar{y}_i)^2+(c-c^0)^2\right]} \tag{13}$$

which is distributed as F with 1 and $n-p+1$ degrees of freedom. This may serve to verify the null hypothesis

$\sum_{i=1}^{p} t_i\beta_i=0$, when $\sum_{i=1}^{p} w_i\beta_i=c^0$ (t_i, w_i and c^0 are given; $i=1,2,\dots,p$;

$$a=\sum_{i=1}^{p}t_i\bar{y}_i,\ b=\sum_{i=1}^{p}\frac{w_i^2}{n_i},\quad c=\sum_{i=1}^{p}w_i\bar{y}_i\quad\text{and}\quad d=\sum_{i=1}^{p}\frac{w_it_i}{n_i};$$

n_i denotes the number of observations, and $\bar{y}_i$ denotes the mean of the i-th class).

The special case of (13) is a known expression for F, which is used when the linear relations $\sum_{i=1}^{p} t_i\beta_i$ and $\sum_{i=1}^{p} w_i\beta_i$ are orthogonal.

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MATHÉMATIQUE

Étude de la matrice des solutions fondamentales du système paraboliques d'équation aux dérivées partielles

par
W. POGORZELSKI

Présenté par T. WAŻEWSKI le 15 novembre 1957

Considérons le système suivant de N équations aux dérivées partielles d'ordre pair M:

$$\hat{\Psi}^{(a)}_{X,t}(u_1, \ldots, u_N) = \sum_{\substack{k_1+\ldots+k_n=M \\ 1\leqslant j\leqslant N}} A^{k_1\ldots k_n}_{aj}(X,t)\frac{\partial^M u_j}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}} + \tag{1}$$
$$+\sum_{\substack{1\leqslant k_1+\ldots+k_n<M \\ 1\leqslant j\leqslant N}} B^{k_1\ldots k_n}_{aj}(X,t)\frac{\partial^{k_1+\ldots k_n} u_j}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}} + \sum_{1\leqslant j\leqslant N} C_{aj}(X,t)u_j - \frac{\partial u_a}{\partial t} = 0$$

où les coefficients $A^{k_1\ldots k_n}_{aj}(X,t)$, $B^{k_1\ldots k_n}_{aj}(X,t)$, $C_{aj}(X,t)$ sont des fonctions complexes des coordonnées réelles du point $X(x_1, \ldots, x_n)$ et de la variable réelle t, définies dans la région

$$X \in E; \quad 0 \leqslant t \leqslant T, \tag{2}$$

E étant un espace euclidien à $n \geqslant 2$ dimensions.

Les solutions fondamentales du système de la forme (1) ont été étudiées par Petrovski [1] dans le cas particulier où les coefficients ne dépendent que de la variable t. Ensuite, ces solutions ont été étudiées par Brouk [2] et récemment par Eidelmann [3] dans le cas où les coefficients dépendent des variables X et t, mais sous l'hypothèse restrictive que ces coefficients admettent les dérivées spatiales de plusieurs ordres.

Je vais présenter dans cette communication les résultats de mes recherches concernant la détermination de la matrice des solutions fondamentales de (1) sous l'hypothèse beaucoup plus générale, à savoir que les coefficients des équations (1), définis dans la région (2) y sont bornés et vérifient les conditions d'Hölder de la forme

$$|A^{k_1\ldots k_n}_{aj}(X,t) - A^{k_1\ldots k_n}_{aj}(X_1,t_1)| < \text{const}[|XX_1|^h + |t-t_1|^{h'}], \tag{3}$$

(3′) $$|B_{aj}^{k_1\ldots k_n}(X,t)-B_{aj}^{k_1\ldots k_n}(X_1,t_1)| < \text{const}\cdot|XX_1|^h,$$

(3″) $$|C_{aj}(X,t)-C_{aj}(X_1,t) < \text{const}\cdot|XX_1|^h,$$

où $|XX_1|$ est la distance euclidienne entre les points X et X_1, les exposants h et h' étant des constantes positives non supérieures à l'unité. La condition d'Hölder par rapport à la variable t pour les coefficients B_{aj} et C_{aj} n'est pas nécessaire; nous n'admettons que leur continuité par rapport à la variable t.

Conformément à la définition du système parabolique, donnée par Petrovski [1], admettons que l'équation

(4) $$\det_{a,j}\Big|\sum_{k_1+\ldots+k_n=M} A_{aj}^{k_1\ldots k_n}(X,t)(is_1)^{k_1}(is_2)^{k_2}\ldots(is_n)^{k_n}-\lambda\delta_a^j\Big| = 0$$

aux coefficients A_{aj}, a les racines en λ dont toutes les parties réelles sont inférieures à un nombre négatif fixé $-\delta$:

(5) $$\mathrm{Re}(\lambda) < -\delta < 0$$

dans la région (2) et pour toutes les valeurs réelles des paramètres $s_1, \ldots, s_n$ vérifiant l'inégalité

(6) $$s_1^2+\ldots+s_n^2 = 1\,.$$

Je me suis appuyé dans mes recherches sur les méthodes exposées dans mon travail [4] et j'ai profité de quelques résultats des travaux [1] et [3], surtout de l'important théorème de Gelfand et Chilov [5] sur les transformations de Fourier des fonctions entières.

J'ai déterminé d'abord la matrice, dite des quasi-solutions, dans la forme

(7) $$W_{a\beta}^{Z,\zeta}(X,t;Y,\tau) = \frac{1}{(2\pi)^n}\int_{-\infty}^{+\infty} v_a^\beta(t,\tau;Z,\zeta;S)\exp\Big[i\sum_{\nu=1}^{n} s_\nu(x_\nu-\xi_\nu)\Big]ds_1\ldots ds_n,$$

où les fonctions $v_a^\beta(t,\tau;Z,\zeta;s_1,\ldots s_n)$ forment les N solutions du système d'équations

(8) $$\frac{dv_a^\beta}{dt} = \sum_{1\leqslant j\leqslant N}^{k_1+\ldots+k_n=M} A_{aj}^{k_1\ldots k_n}(Z,\zeta)(is_1)^{k_1}\ldots(is_n)^{k_n}v_j^\beta(t,\tau;Z,\zeta;S)$$

vérifiant la condition initiale $v_a^\beta = \delta_a^\beta$ si $t=\tau$; $Y(\xi_1,\ldots\xi_n)$, Z,ζ,τ sont des paramètres. J'ai établi ensuite les théorèmes suivants.

Théorème 1. *Si les coefficients A_{aj} sont bornés et continus dans la région* (2), *l'hypothèse* (5) *étant vérifiée et si en outre les fonctions $\varrho_\beta(Y,\tau)$, supposées bornées dans la région* (2) *et intégrables dans toute la partie me-*

surable de cette région, sont continues dans un domaine $\Omega \subset E$ pour $0 \leqslant \tau \leqslant T$, alors les fonctions

$$(9) \qquad J_\alpha(X, \tau, t) = \iiint_E \sum_{\beta=1}^{N} W_{\alpha\beta}^{Y,\tau}(X, t; Y, \tau)\, \varrho_\beta(Y, \tau)\, dY$$

où $0 \leqslant \tau < t \leqslant T$, analogues à l'intégrale de Poisson-Weierstrass pour l'équation de chaleur, ont la propriété limite suivante:

$$(10) \qquad \lim_{\tau \to t} J_\alpha(X, \tau, t) = \varrho_\alpha(X, t)$$

en tout point intérieur $X \in \Omega$.

THÉORÈME 2. *Si les fonctions $\varrho(Y, \tau)$ sont bornées dans la région* (2) *et intégrables dans toute sa partie mesurable, les fonctions*

$$(11) \qquad V_\alpha(X, t) = \int_0^t \iiint_E \sum_{\beta=1}^{N} W_{\alpha\beta}^{Y,\tau}(X, t; Y, \tau)\, \varrho_\beta(Y, \tau)\, dY d\tau,$$

dites composantes du quasi-potentiel de charge spatiale, admettent les dérivées spatiales continues d'ordre $m \leqslant M-1$ données par les intégrales absolument convergentes

$$(12) \qquad D_X^{(m)}[V_\alpha(X, t)] = \int_0^t \iiint_E \sum_{\beta=1}^{N} D_X^{(m)}[W_{\alpha\beta}^{Y,\tau}(X, t; Y, \tau)]\, \varrho_\beta(Y, \tau)\, dY d\tau .$$

THÉORÈME 3. *Si les conditions* (3) *et* (5) *sont satisfaites et si les fonctions $\varrho_\beta(Y, \tau)$ bornées et continues dans* (2), *vérifient la condition d'Hölder*

$$(13) \qquad |\varrho_\beta(X, t) - \varrho_\beta(X_1, t)| < \text{const.}\, |XX_1|^r \qquad (0 < r \leqslant 1)$$

dans la région $[X \in \Omega;\ 0 \leqslant t \leqslant T]$, où Ω est un domaine mesurable dans l'espace E, les composantes du quasi-potentiel (11) *admettent en tout point intérieur $X \in \Omega$ pour $0 < t < T$ les dérivées spatiales d'ordre M représentées par la formule*

$$(14) \qquad D_X^{(M)}[V_\alpha(X, t)] = \int_0^t \iiint_E \sum_{\beta=1}^{N} D_X^{(M)}[W_{\alpha\beta}^{Y,\tau}(X, t; Y, \tau)]\, \varrho_\beta(Y, \tau)\, dY d\tau .$$

Les intégrales dans cette formule ne sont pas, en général, absolument convergentes, mais les intégrales spatiales, qui sont les dérivées d'ordre M de l'intégrale (9), admettent la limitation à singularité faible

$$(15) \qquad |D_X^{(M)}[J(X, t, \tau)]| < \frac{C_X}{(t-\tau)^{\mu}},$$

où μ est une constante fixée arbitrairement dans l'intervalle ouvert $1 - h_*/M < \mu < 1$, $h_* = \min(h, r)$, et C_X étant une constante fixée au voisinage du point X.

THÉORÈME 4. *Si les conditions du théorème 3 sont satisfaites, les composantes du quasi-potentiel* (11) *vérifient les équations suivantes:*

$$(16)\quad \sum_{1\leqslant j\leqslant N}^{k_1+\ldots+k_n=M} A_{aj}^{k_1\ldots k_n}(X,t)\frac{\partial^M V_j(X,t)}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}} - \frac{\partial}{\partial t}[V_a(X,t)] = -\varrho_a(X,t)+$$

$$+\int_0^t\iiint_E \sum_{1\leqslant\beta,j\leqslant N}^{k_1+\ldots+k_n=M}[A_{aj}^{k_1\ldots k_n}(X,t)-A_{aj}^{k_1\ldots k_n}(Y,\tau)]\frac{\partial^M W_j^{Y,\tau}(X,t;Y,\tau)}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}}\varrho_\beta(Y,\tau)dYd\tau$$

$(\alpha=1,2,\ldots,N)$ *en tout point intérieur* $X\epsilon\Omega$ *et pour* $0<t<T$.

THÉORÈME 5. *Si les conditions* (3), (3′), (3″) et (5) *sont satisfaites, il existe N solutions fondamentales* $\Gamma_{1\beta},\Gamma_{2\beta},\ldots,\Gamma_{N\beta}$ $(\beta=1,2,\ldots,N)$ *du système* (1), *données par les formules*

$$(17)\quad \Gamma_{a\beta}(X,t;Y,\tau)=$$
$$= W_{a\beta}^{Y,\tau}(X,t;Y,\tau)+\int_\tau^t\iiint_E\sum_{\gamma=1}^N W_{a\gamma}^{\Pi,\zeta}(X,t;\Pi,\zeta)\Phi_{\gamma\beta}(\Pi,\zeta;Y,\tau)d\Pi d\zeta$$

où $X(x_1,\ldots,x_n)\neq Y(\xi_1,\ldots\xi_n)$ *sont deux points arbitraires de l'espace E et* $0<\tau<t<T$.

Les fonctions $\Phi_{\gamma\beta}(\Pi,\zeta;Y,\tau)$ sont des solutions du système d'équations de Volterra

$$(18)\quad \Phi_{a\beta}(X,t;Y,\tau)=\hat\Psi_{(j)X,t}^{(a)}[W_{j\beta}^{Y,\tau}(X,t;Y,\tau)]+$$
$$+\int_\tau^t\iiint_E\sum_{\gamma=1}^N\hat\Psi_{(j)X,t}^{(a)}[W_{j\gamma}^{\Pi,\zeta}(X,t;\Pi,\zeta)]\Phi_{\gamma\beta}(\Pi,\zeta;Y,\tau)d\Pi d\zeta$$

où $\alpha,\beta=1,2,\ldots,N$ et $\Psi_{(j)}^{(a)}$ désignent les opérateurs différentiels (1

Il a été prouvé au cours de la démonstration que les solutions du système (18) vérifient les inégalités

$$(19)\quad |\Phi_{a\beta}(X,t;Y,\tau)|\leqslant \mathrm{const}\,(t-\tau)^{-\mu}/|XY|^{n+M(1-\mu)-h_1}$$

lorsque $|XY|\leqslant\varrho_0$ et $1-h_1/M<\mu<1$ où $h_1=\min(h,2h')$ et les inégalités

$$(19')\quad |\Phi_{a\beta}(X,t;Y,\tau)|<\mathrm{const}\,(t-\tau)^{\mu_1}/|XY|^{n+M\mu_1}\qquad \text{où}\qquad \mu_1>0,$$

lorsque $|XY|>\varrho_0$.

En outre, il est important de signaler, que les fonctions $\Phi_{a\beta}$ vérifient la condition d'Hölder (voir [4])

$$(20)\quad |\Phi_{a\beta}(X,t;Y,\tau)-\Phi_{a\beta}(X_1,t;Y,\tau)|<\frac{\mathrm{const}}{\inf|XY|^{n+M+1}}|XX_1|^{\theta h_1}$$

pour $0<\theta<1$ en tout domaine fermé Ω^* de l'espace E ne contenant pas le point Y.

Les démonstrations des théorèmes 1—5 seront publiées dans Ricerche di Matematica (Napoli, 1958).

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MATHÉMATIQUE

Propriétés d'une intégrale singulière pour les arcs non fermés et leurs application

par

W. POGORZELSKI

Présenté par T. WAŻEWSKI le 16 novembre 1957

Je me propose de présenter dans cette communication quelques résultats de mes recherches concernant l'intégrale singulière de la forme

$$F(t) = \int_L \frac{f(t,\tau)\,d\tau}{\tau - t} \tag{1}$$

et son application à l'équation intégrale non linéaire singulière. L y désigne un ensemble composé d'un nombre fini p d'arcs dirigés $a_1b_1, a_2b_2, \ldots, a_pb_p$ disjoints, ayant les extrémités différentes, situés sur le plan de la variable complexe; t et τ désignent deux points variables sur ces arcs; l'intégrale (1) a le sens de la valeur principale de Cauchy.

Admettons les hypothèses suivantes:

(I) La fonction complexe $f(t,\tau)$ est définie dans la région

$$t \in L; \quad \tau \in L$$

(les extrémités des arcs $a_\nu b_\nu$ exceptées); elle y est continue et vérifie les inégalités

$$|f(t,\tau)| < \frac{M_f}{\prod_{\nu=1}^{p} |\tau - a_\nu|^\beta |\tau - b_\nu|^\beta}, \tag{2}$$

$$|f(t,\tau) - f(t_1,\tau_1)| < \frac{k_f}{[\prod_{\nu=1}^{p} |\tau - a_\nu|\;|\tau_1 - b_\nu|]^{\alpha+\mu}} [|t - t_1|^{\mu_1} + |\tau - \tau_1|^{\mu}], \tag{3}$$

où M_f et k_f sont des constantes positives données tandis que les constantes données $\alpha, \beta, \mu, \mu_1$ vérifient les conditions

$$0 < \beta < \alpha < 1; \quad 0 < \mu < \mu_1 \leqslant 1; \quad \alpha + \mu < 1. \tag{4}$$

Il est supposé en outre que le point t_1 suit le point t et que le point τ_1 suit le point τ dans la direction de l'arc $a_\nu b_\nu$ lorsque les points du couple (t, t_1) ou du couple (τ, τ_1) se trouvent sur le même arc $a_\nu b_\nu$.

(II) Les arcs formant L admettent la tangente continue en tout point; en outre, l'angle $\vartheta(t, t_1)$ en deux points arbitraires t, t_1 des arcs $a_\nu T'_\nu$, $T''_\nu b_\nu$ respectivement, qui sont des arcs partiels de l'arc $a_\nu b_\nu$ $(\nu = 1, 2, \ldots p)$ issus de ses extrémités, vérifie l'inégalité

$$\vartheta(t, t_1) < \text{const.}\ |t-t_1|^h \tag{5}$$

où $\mu < h \leqslant 1$.

J'ai démontré deux théorèmes suivants.

Théorème 1. *Si les hypothèses* (I) *et* (II) *sont satisfaites, l'intégrale singulière* (1) *vérifie la limitation suivante*:

$$|F(t)| < \frac{CM_f + C'k_f}{\prod_{\nu=1}^{p} |t-a_\nu|^a\, |t-b_\nu|^a} \tag{6}$$

en tout point $t \in L$ *différent des extrémités* a_ν, b_ν; C *et* C' *sont des constantes positives ne dépendant que des arcs de* L.

Théorème 2. *Si les hypothèses* (I) *et* (II) *sont satisfaites, l'intégrale* (1) *vérifie la condition d'Hölder de la forme*

$$|F(t)-F(t_1)| < \frac{C_1 M_f + C'_1 k_f}{[\prod_{\nu=1}^{p} |t-a_\nu| \cdot |t_1-b_\nu|]^{a+\mu}}\, |t-t_1|^\mu, \tag{7}$$

où le point t_1 *suit le point* t *lorsque les deux points sont sur le même arc* $a_\nu b_\nu$; C_1 *et* C'_1 *sont des constantes positives ne dépendant que des arcs* L.

En s'appuyant sur les théorèmes 1 et 2 et en appliquant le théorème topologique de Schauder sur le point invariant d'une transformation dans l'espace de Banach, j'ai démontré l'existence de la solution d'une équation intégrale singulière de la forme

$$\varphi(t) = \lambda \int_L \frac{F[t, \tau, \varphi(\tau)]\, d\tau}{\tau - t}. \tag{8}$$

Les équations intégrales singulères non linéaires du type indiqué sont traitées dans les travaux [1]—[6].

On a le théorème suivant:

Théorème 3. *Si la fonction* $F(t, \tau, u)$, *définie dans la région*

$$t \in L; \qquad \tau \in L; \qquad u \in \Pi \tag{9}$$

(Π désignant le plan de la variable complexe) vérifie les inégalités

$$|F(t, \tau, u)| < m_F |u|^r + m'_F, \tag{10}$$

$$|F(t, \tau, u) - F(t_1, \tau_1, u_1)| < k_F [|t - t_1|^{\mu_1} + |\tau - \tau_1|^{\mu} + |u - u_1|], \tag{11}$$

où les constantes positives données m_F, m'_F, r, k_F, μ_1, μ satisfont aux conditions:

$$0 < \mu < \mu_1 \leq 1; \quad 0 < r < 1; \quad \mu + r \leq 1; \quad \mu < h < 1, \tag{12}$$

il existe au moins une solution $\varphi(t)$ de l'équation (8) continue à l'intérieur des arcs formant L, pourvu que le module du parametre λ soit suffisamment petit, à savoir satisfaisant à l'inégalité de la forme $|\lambda| k_f < C$, où C est une constante positive ne dépendant que de la forme géométrique des arcs formant L et de l'exposant r.

La démonstration des théorèmes 1—3 sera publiée dans Journal of Mathematics and Mechanics.

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MATHEMATICS

A Generalization of a Theorem Concerning the Power of a Perfect Compact Metric Space

by

S. MRÓWKA

Presented by W. SIERPIŃSKI on November 30, 1957

The following theorem is well-known:

(A): *If X is a perfect compact metric space, then* $\overline{\overline{X}} = 2^{\aleph_0}$.

Using the notion of the character of a point (the character of a point p is, by definition, the least cardinal $\mathfrak{m}$ for which there is a basis of neighbourhoods of the point p which has the cardinality $\mathfrak{m}$), we can formulate this theorem as follows (notice that a point of a metric space is isolated if, and only if, it is of the character 1 and is non-isolated if, and only if, it is of the character $\aleph_0$):

(A′): *If each point of a compact metric space X is of the character $\aleph_0$, then* $\overline{\overline{X}} = 2^{\aleph_0}$.

On the other hand, in [2] the author proved the following theorem:

(B): *If X is a compact space* *) *containing no point satisfying the first axiom of countability, then* $\overline{\overline{X}} \geqslant 2^{\aleph_0}$.

This theorem can be formulated in the following way:

(B′): *If each point of a compact space X is of the character $\geqslant \aleph_1$, then* $\overline{\overline{X}} \geqslant 2^{\aleph_0}$.

In this paper we give a generalization of Theorems (A′) and (B′).

Given a topological space S, and a cardinal $\mathfrak{m}$ **), we denote as $\mathfrak{A}_\mathfrak{m}(S)$ the family of all closed non-empty subsets of S which are the intersections of at most $\mathfrak{m}$ open sets and let $\mathfrak{B}_\mathfrak{m}(S) = \bigcup_{\mathfrak{n}<\mathfrak{m}} \mathfrak{A}_\mathfrak{n}(S)$. Of course, we have:

(i) $$\mathfrak{A}_\mathfrak{n}(S) \subset \mathfrak{A}_\mathfrak{m}(S) \quad \text{for} \quad \mathfrak{n} < \mathfrak{m};$$

(ii) $$\text{if} \quad T \in \mathfrak{A}_\mathfrak{m}(S), \quad \text{then} \quad \mathfrak{A}_\mathfrak{m}(T) \subset \mathfrak{A}_\mathfrak{m}(S).$$

*) We use the term "compact" in the sense of Bourbaki (= "bicompact" in the sense of Alexandroff and Urysohn).

**) In this paper $\mathfrak{m}$ is supposed to be an infinite cardinal.

We shall show the following

LEMMA. *If each point of a compact space S is of the character $\geqslant\mathfrak{m}$, where $\mathfrak{m}$ is a regular cardinal, then there is a family $\{A^{(\varphi)}\}_{\varphi\in\Phi}$ $(\overline{\overline{\Phi}}=2^{\mathfrak{m}})$ of mutually disjoint membres of $\mathfrak{A}_{\mathfrak{m}}(S)$.*

Proof *). The pattern of proof is as follows: let $\mathfrak{m}=\aleph_a$ and denote as Φ the class of all transfinite sequence of 0's and 1's of the type ω_a. We define a class $\mathfrak{B}$ of members of $\mathfrak{A}_{\mathfrak{m}}(S)$ which has the following properties:

1^0 if $B_1, B_2, \ldots, B_\xi, \ldots$ is any decreasing sequence of members of $\mathfrak{B}$ of the type $<\omega_a$, then $\bigcap\limits_{\xi} B_\xi\in\mathfrak{B}$;

2^0 there are two functions $\mathfrak{f}_0$ and $\mathfrak{f}_1$ defined on $\mathfrak{B}$ and with values from $\mathfrak{B}$ such that $\mathfrak{f}_0(B)\cup\mathfrak{f}_1(B)\subset B$ and $\mathfrak{f}_0(B)\cap\mathfrak{f}_1(B)=0$ for each B in $\mathfrak{B}$.

If the class $\mathfrak{B}$ is defined, then we assign to each sequence $\varphi=\{\beta_1,\beta_2,\ldots,\beta_\xi,\ldots\}_{\xi<\omega_a}$ in Φ a sequence $\{B_1^{(\varphi)}, B_2^{(\varphi)},\ldots,B_\xi^{(\varphi)},\ldots\}_{\xi<\omega_a}$ of members of $\mathfrak{B}$ defined, by transfinite induction, in the following manner:

1. $B_1=\mathfrak{f}_{\beta_1}(S)$.

2. Suppose that for some $\xi_0<\omega_a$ the sets $B_1,\ldots,B_\xi,\ldots$ are defined for each $\xi<\xi_0$, they are members of $\mathfrak{B}$ and they form a decreasing sequence. In virtue of 1^0, $B=\bigcap\limits_{\xi<\xi_0}B_\xi^{(\varphi)}$ is a member of $\mathfrak{B}$. Let $B_{\xi_0}^{(\varphi)}=\mathfrak{f}_{\beta_{\xi_0}}(B)$. Then $B_\xi^{(\varphi)}$ is a member of $\mathfrak{B}$, and, clearly, $B_\xi^{(\varphi)}\subset B_{\xi_0}^{(\varphi)}$ for each $\xi<\xi_0$, hence the sequence $B_1^{(\varphi)}, B_2^{(\varphi)},\ldots,B_{\xi_0}^{(\varphi)}$ is also decreasing.

It can be easily seen, that if $\varphi=\{\beta_1,\beta_2,\ldots,\beta_\xi,\ldots\}_{\xi<\omega_a}$, $\varphi'=\{\beta_1',\beta_2',\ldots,\beta_\xi',\ldots\}_{\xi<\omega_a}$ are members of Φ and $\beta_{\xi_0}\neq\beta_{\xi_0}'$, then $B_{\xi_0}^{(\varphi)}\cap B_{\xi_0}^{(\varphi')}=0$. Hence, setting for each φ in Φ, $A^{(\varphi)}=\bigcap\limits_{\xi<\omega_a}B_\xi^{(\varphi)}$, we have $A^{(\varphi)}\cap A^{(\varphi')}=0$ for each two distinct membres φ and φ' of Φ. Moreover, the set $A^{(\varphi)}$, as the intersection of a decreasing sequence of the type ω_a of members of $\mathfrak{A}_{\mathfrak{m}}(S)$ is a member of $\mathfrak{A}_{\mathfrak{m}}(S)$. Thus $\{A^{(\varphi)}\}_{\varphi\in\Phi}$ is the required family.

It remains to define the class $\mathfrak{B}$. We shall distinguish two cases:

1st case: $\mathfrak{m}=\aleph_0$. Then, we take as $\mathfrak{B}$ the class of all members of $\mathfrak{A}_{\mathfrak{m}}(S)$ which have the non-empty interior. Clearly, the property 1^0 is satisfied. To prove the second, let B be any member of $\mathfrak{B}$. Since no point of S is isolated, the interior of B contains at least two distinct points, say p_0 and p_1. Let f be a continuous function on the space S to the unit interval $[0,1]$ such that $f(p)_0=0$ and $f(p)_1=1$. Setting $B_0=\{p\in B;\ f(p)\leqslant\frac{1}{3}\}$ and $B_1=\{p\in B;\ f(p)\geqslant\frac{2}{3}\}$ we have $B_0\cap B_1=0$, $B_0\cup B_1\subset B$ and B_0, B_1 are members of $\mathfrak{B}$. Hence, using the axiom of choice, it is easy to define the functions $\mathfrak{f}_0$ and $\mathfrak{f}_1$.

*) The idea of this proof is similar to that of the proof given in [2].

2nd case: $\mathfrak{m} > \aleph_0$. Now we take as $\mathfrak{B}$ the family $\mathfrak{B}_{\mathfrak{m}}(S)$. Clearly, .e property 1^0 is satisfied and we shall show that the second is also. ət us notice that, in virtue of the Alexandroff-Urysohn theorem ating that in a compact space the pseudocharacter of each point co- cides with its character (see [1], chapter II, § 2, Theorem 4; the pseudo- aracter of a point p is the least cardinal $\mathfrak{n}$ for which there exists a fa- ily $\mathfrak{U}$ of neighbourhoods of p with $\overline{\overline{\mathfrak{U}}} = \mathfrak{n}$ and such that the total inter- ction of $\mathfrak{U}$ contains only the point p), each member of $\mathfrak{B}$ contains at ast two distinct points. Let $\mathfrak{B}$ be any member of B and let p_0 and p distinct points of B. There is a continuous function f on S to the unit terval $[0, 1]$ such that $f(p_0) = 0$ and $f(p_1) = 1$. Let $B_0 = B \cap f^{-1}(0)$ ıd $B_1 = B \cap f^{-1}(1)$. Then the sets B_0 and B_1 are members of $\mathfrak{B}$ (the ts $f^{-1}(0)$ and $f^{-1}(1)$ are the countable intersections of open sets). $_0 \cap B_1 = 0$, $B_0 \cup B_1 \subset B$. Using the axiom of choice, we can define the nctions $\mathfrak{f}_0$ and $\mathfrak{f}_1$. Thus the proof of the Lemma is over.

THEOREM 1. *If each point of a compact space X is of the character $\geqslant \mathfrak{m}$, en $\overline{\overline{X}} \geqslant 2^{\mathfrak{m}}$.*

Proof. If $\mathfrak{m}$ is a regular cardinal, then Theorem 1 immediately llows from the preceding Lemma. Thus, one can assume that $\mathfrak{m}$ is an regular cardinal. Let $\mathfrak{m} = \aleph_\alpha$, where α is a limit ordinal, with $\overline{\alpha} < \mathfrak{m}$. hen $\mathfrak{m} = \sum_{\xi<\alpha} \aleph_\xi$. Of course, we have also $\mathfrak{m} = \sum_{\xi<\alpha} \aleph_{\xi+1}$. Let Φ_ξ $(\xi < \alpha)$ any set of the power $2^{\aleph_{\xi-1}}$ and let Ψ be the set of all sequences $_1, \ldots, \varphi_\xi, \ldots\}_{\xi<\alpha}$ where $\varphi_\xi \in \Phi_\xi$. Clearly, $\Psi = \prod_{\xi<\alpha} 2^{\aleph_{\xi+1}} = 2^{\sum_{\xi<\xi_0} \aleph_{\xi+1}} = 2^{\mathfrak{m}}$.

Let A be any member of $\mathfrak{A}_{\aleph_{\xi+1}}(X)$. By the quoted result of Alexandroff ıd Urysohn, the space A satisfies all the assumptions of the Lemma $_{\xi+1}$ is a regular cardinal), hence there is a family $\{A^{(\varphi)}\}_{\varphi\in\Phi_\xi}$ of mu- ally disjoint members of $\mathfrak{A}_{\aleph_{\xi+1}}(A) \subset \mathfrak{A}_{\aleph_{\xi+1}}(X)$. Using the axiom of ıoice, one can define a family $\{\mathfrak{f}_\varphi^{(\xi)}\}_{\varphi\in\Phi_\xi}$ of functions, each on $\mathfrak{A}_{\aleph_{\xi+1}}(X)$ $\mathfrak{A}_{\aleph_{\xi+1}}(X)$, such that $\mathfrak{f}_\varphi^{(\xi)}(A) \subset A$, $\mathfrak{f}_\varphi^{(\xi)}(A) \cap \mathfrak{f}_{\varphi'}^{(\xi)}(A) = 0$ for each A in $_{\xi+1}(X)$ and every φ, φ' in Φ_ξ with $\varphi \neq \varphi'$.

Let $\psi = \{\varphi_1, \varphi_2, \ldots, \varphi_\xi, \ldots\}_{\xi<\alpha}$ be any member of Ψ. We define, by .e transfinite induction, the sequence $\{A_1^{(\psi)}, \ldots, A_\xi^{(\psi)}, \ldots\}_{\xi<\alpha}$ in the fol- wing manner:

1. $A_1^{(\psi)} = f_{\varphi_1}^{(1)}(X)$.

2. Suppose that for some $\xi_0 < \alpha$, the sets $A_1^{(\psi)}, \ldots, A_\xi^{(\psi)}, \ldots$ are de- ıed for each $\xi > \xi_0$, they form a decreasing sequence and $A_\xi^{(\psi)} \in A_{\aleph_{\xi+1}}(X)$. ət $A = \bigcap_{\xi<\xi_0} A_\xi^{(\psi)}$. Then A is a closed non-empty subset of X. On the her hand, A is the intersection of at most $\sum_{\xi<\xi_0} \aleph_{\xi+1} \leqslant \aleph_{\xi_0+1}$ [open sets,

hence $A \in \mathfrak{A}_{\aleph_{\xi+1}}(X)$. Let $A^{(\psi)}_{\xi_0} = \mathfrak{f}^{(\xi_0)}_{\varphi_{\xi_0}}(A)$. Then $A^{(\psi)}_{\xi_0}$ is a member of $\mathfrak{A}_{\aleph_{\xi_0+1}}(X)$ and the sets $A^{(\psi)}_1, \ldots, A^{(\psi)}_{\xi_0}$ form a decreasing sequence. Thus the sequence $\{A^{(\psi)}_1, \ldots, A^{(\psi)}_{\xi}, \ldots\}_{\xi<\alpha}$ is defined.

Let $A^{(\psi)} = \bigcap_{\xi<\alpha} A^{(\psi)}_{\xi}$. Then $A^{(\psi)}$ is a non-empty subset of X and if ψ, ψ' are distinct members of Ψ, then $A^{(\psi)} \cap A^{(\psi')} = 0$. It follows that the power of X is not less than of the set Ψ, i. e. $\overline{\overline{X}} \geqslant 2^{\mathfrak{m}}$. Thus the proof of Theorem 1 is over.

If follows from the above proof that our Lemma is valid if $\mathfrak{m}$ is any infinite cardinal (not necessarily regular).

Remark. In the case $\mathfrak{m} = \aleph_0$, Theorem 1 cannot be reduced to the Theorem (A) (or (A′)). In fact, there exists a non-metrizable compact space whose each point is of the character $\aleph_0$ (see, for instance, [1], chapter V, § 1, the space A_7).

Now we give some consequences of Theorem 1.

THEOREM 2. *If X is a compact $\mathfrak{m}$-almost-metrizable space* *) *and each point of X is of the character $\mathfrak{m}$, then $\overline{\overline{X}} = 2^{\mathfrak{m}}$.*

Proof. It suffices to show the inequality $\overline{\overline{X}} \leqslant 2^{\mathfrak{m}}$. By Theorem 5 of [3], X contains a dense subset X_0 of the power $\leqslant \mathfrak{m}$. By Theorem 3 of [3], each point of X is the limit of an $\mathfrak{m}$-sequence of points of X_0. But the power of the set of all $\mathfrak{m}$-sequences of points of X_0 is not greater than $2^{\mathfrak{m}}$ and it follows that $X \leqslant 2^{\mathfrak{m}}$.

THEOREM 3. *If X is a locally compact space and each point of X is of the character $\geqslant \mathfrak{m}$, then $\overline{\overline{X}} \geqslant 2^{\mathfrak{m}}$.*

Proof. It suffices to show that there exists a compact set $A \subset X$ such that each point of A is of the character $\geqslant \mathfrak{m}$ with respect to A. Let p_0 be any point of X and let U be a neighbourhood of p_0 having the compact closure. We shall distinguish two cases:

a) $\mathfrak{m} = \aleph_0$. Then we take as A the set $\overline{U}$. Since X is a closed domain and no point of X is isolated, no point of A is isolated with respect to A; i. e. each point of A is of the character $\geqslant \aleph_0$ with respect to A.

b) $\mathfrak{m} > \aleph_0$. There is a continuous function f on X to the unit interval $[0, 1]$ which is one at p_0 and zero on the complement of U. Let us set $A = f^{-1}(1)$. Clearly, A is compact. On the other hand, since A is a G_δ-set, we infer, by the above mentioned Alexandroff-Urysohn Theorem, that each point of A is of the character $\geqslant \mathfrak{m}$ with respect to A.

An immediate consequence of Theorem 3 is the following

COLLORARY 1 (Hulanicki [5]). *If G is a locally compact topological group and the unit element of G is of the character $\geqslant \mathfrak{m}$, then $\overline{\overline{G}} \geqslant 2^{\mathfrak{m}}$.*

*) The definition and the fundamental properties of almost-metric spaces are given in [3].

It can be showed that if the unit element of a locally compact group is of the character $\mathfrak{m}$, then the group is $\mathfrak{m}$-almost-metrizable. Hence, we have

COROLLARY 2. *If G is a locally compact topological group and $\mathfrak{m}$ is the least cardinal for which the group G is $\mathfrak{m}$-almost-metrizable, then $\overline{\overline{G}} \geqslant 2^{\mathfrak{m}}$.*

From Theorem 2 we obtain:

COROLLARY 3. *If G is a compact topological group and the unit element of G is of the character $\mathfrak{m}$, then $\overline{\overline{G}} = 2^{\mathfrak{m}}$.*

Thus, the power of a compact topological group is uniquely determined by the character of its unit element.

Remark. Theorem 1 gives the lower estimation of the power of a compact topological space. The problem arises, whether the converse estimation is also valid. More exactly, suppose X is a compact topological space and each point of X is of the character $\leqslant \mathfrak{m}$; is the space X of the power $\leqslant 2^{\mathfrak{m}}$? This problem, in the particular case $\mathfrak{m} = \aleph_0$ was raised by Alexandroff and Urysohn (see [4], end of paper), but as far as I know, this problem is still unsolved. On the other hand, Alexandroff and Urysohn have proved that if in a compact space each closed subset is a G_δ, then the space is of the power $\leqslant 2^{\aleph_0}$ (see [1], chapter II, § 3, Theorem 6). By the same argument as that used by Alexandroff and Urysohn the following statement can be established:

If X is a compact space and each closed subsed of X is the intersection of at most $\mathfrak{m}$ open subsets, then $X \leqslant 2^{\mathfrak{m}}$.

At the end we give a problem related to the topic of the paper.

PROBLEM. *If X is an uncountable compact metric space, then we can show, without the hypothesis of the continuum, that $\overline{\overline{X}} = 2^{\aleph_0}$. Can we prove, without the generalized hypothesis of the continuum, the following statement:*

If X is a compact $\mathfrak{m}$-almost-metrizable space of the power $> \mathfrak{m}$, then $\overline{\overline{X}} = 2^{\mathfrak{m}}$?

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MATHEMATICS

A Property of Hewitt Extension νX of Topological Spaces

by

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Presented by K. KURATOWSKI on December 2, 1957

It is known that the Stone-Čech extension βX of topological spaces can be characterized by the following conditions:

a) *βX is a compact space and it contains X as a dense subspace;*

b) *each function $f \in I^X$ admits an extension $f^* \in I^{\beta X}$* *).

Similarly, the Hewitt extension νX [1] can be characterized by the conditions:

1) *νX is a Q-space and it contains X as a dense subspace*;

2) *each function $f \in R^X$ admits an extension $f^* \in R^{\nu X}$.*

On the other hand, it is known that 1) remains valid if I is replaced by any compact space. We shall show that 2) is valid if R is replaced by any Q-space; namely, we shall prove the following:

THEOREM: *If Y is any Q-space, then each function $f \in Y^X$ admits an extension $f \in Y^{\nu X}$.*

Proof. Suppose $f \in Y^X$. Since each continuous real-valued function defined on X can be extended over νX, νX may be regarded as a subset of βX. Clearly, f admits an extension $f^* \in \beta Y^{\beta X}$; it suffices to show that if $x \in \nu X$, then $f^*(x) \in Y$. Suppose $y_0 = f^*(x_0) \in \beta Y \setminus Y$. Since Y is a Q-space, there is a continuous real-valued function φ defined on βY with $\varphi(y_0) = 0$ and $\varphi(y) > 0$ for $y \in Y$ (see [2], Proposition). Let $\psi(x) = \varphi(f(x))$ for $x \in \beta X$ and $\psi_1 = \psi|X$ **). Since ψ_1 is positive, $1/\psi_1$ is continuous on X, hence $1/\psi_1$ can be continuously extended over νX. It follows that $\psi(x) > 0$ for $x \in \nu X$. But $\psi(x_0) = 0$, consequently $x_0 \bar{\in} \nu X$ and the theorem follows.

Note. This theorem admits a generalization. Suppose E is any completely regular space having the following property: if X is any completely regular space and $x_0 \bar{\in} A = \bar{A} \subset X$, then there is $f \in E^X$ with

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*) Y^X denotes the set of all continuous functions which map X into Y; I denotes the unit closed interval [0.1], R — the real line.

**) $f|A$ denotes the function f restricted to the set A.

$f(x_0) \in \overline{f(A)}$. A space Y is said to be *E-compact* provided that there is no space Y^* containing Y as a dense subspace and such that each $f \in E^Y$ admits an extension $f^* \in E^{Y^*}$. (For example, a space Y is I-compact if, and only if, Y is compact, Y is R-compact if, and only if, Y is a Q-space). Then, for each completely regular space X there is a unique (within homeomorphisms) extension $\nu_E X$ having the following properties:

1^0 *$\nu_E X$ is an E-compact space and it contains X as dense subspace;*

2^0 *each function $f \in E^X$ admits an extension $f^* \in E\, \nu_E X$. (For example, $\nu_I X = \beta X$, $\nu_R X = \nu X$).*

The extension $\nu_E X$ may be defined in various ways; i. e. we may set $\nu_E X = \beta X \setminus X_0$, where X_0 is the set of all points $x_0 \in \beta X$ for which there is $f \in \beta E^{bX}$ with $f(x_0) \in \beta E \setminus E$ and $f(x) \in E$ for $x \in X$.

The following statement holds true:

If Y is any E-compact space, then each function $f \in Y^X$ admits an extension $f^* \in Y\, \nu_E X$.

Many other theorems on extensions βX and νX are valid for arbitrary $\nu_E X$ extensions, for example, *a closed subset of an arbitrary E-compact space in an E-compact space* *). An examination of all these theorems may be of interest.

We get an extension $\nu_E X$, different from βX and νX, if we put as E the "transfinite line" R_1 obtained by taking the product $Z_{\omega_1} \times I$, where Z_{ω_1} is the space of all ordinals $\xi < \omega_1$ and identifying point $(\xi, 1)$ with the point $(\xi+1, 0)$ for each $\xi < \omega_1$.

PROBLEM: *Find the conditions on E_1 and E_2 under which $\nu_1 X \subset \nu_2 X$ holds for any completely regular space X.*

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*) This result is due to R. Engelking.

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MATHEMATICS

A Note on Universal Banach Spaces of a Finite Dimensions

by
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Presented by S. MAZUR on December 5, 1957

Definition 1. A B-space X is called *universal* (with respect to isometric mappings) for a class $\mathfrak{K}$ of B-spaces if, for every space $Y \in \mathfrak{K}$, X contains an isometric image of Y.

In this paper the following theorem is proved *):

THEOREM 1. *There exists no finite dimensional B-space, universal for all two dimensional B-spaces.*

More precisely:

THEOREM 2. *Let n be an arbitrary positive integer. No n-dimensional B-space is universal for all such two dimensional B-spaces, spheres of which are convex $(2n+2)$-gons.*

Definition 2. An n-dimensional centrally symmetric convex body Q is called *universal* for a class $\mathfrak{K}$ of m-dimensional $(m < n)$ convex bodies if, for each body $\Delta \in \mathfrak{K}$, there is an m-dimensional central intersection of Q, which is affinely equivalent to Δ.

Theorem 2 has the following geometrical interpretation:

THEOREM 2′. *Let n be an arbitrary positive integer. No n-dimensional centrally symmetric convex body is universal for all centrally symmetric convex $(2n+2)$-gons.*

It is interesting to note that an *n-dimensional cube is universal for all the centrally symmetric convex $2n$-gons.*

This is a well-known result of Minkowski's geometry.

This paper contains an *a contrario* proof of Theorem 2′: The existence of an n-dimensional body universal for all centrally symmetric convex $(2n+2)$-gons makes it possible to define functions satisfying the

*) This theorem is a solution of problem 41 proposed by S. Mazur in the Scottish Book.

Lipschitz condition, which map a subset of an $(n-1)$-dimensional Euclidean sphere onto a $(2n-2)$-dimensional Euclidean cube. This is impossible (see the following section).

A lemma on Lipschitz mappings

LEMMA. *Let m be a fixed positive integer. Suppose functions $f_1(t)$, $f_2(t), \ldots, f_m(t)$ map a subset Z of a two-dimensional Euclidean sphere into the four-dimensional Euclidean space E^4. If $f_1(Z) \cup f_2(Z) \cup \ldots \cup f_m(Z) \supset K$, where K is a four-dimensional cube, then at least one of these functions does not satisfy the Lipschitz condition.*

Proof. Let us consider a sequence of systems of points belonging to K *)

$$s_1^{(n)}, s_2^{(n)}, \ldots, s_{(m+1)n^4}^{(n)}$$

such that

$$|s_i^{(n)} s_j^{(n)}| \geqslant \frac{\operatorname{diam} K}{n(m+1)} \quad i \neq j,\ n = 1, 2, \ldots) \text{**)}. \tag{1}$$

Now for any natural n, let us decompose the set Z into n^4 disjoint parts $Z_1^{(n)}, Z_2^{(n)}, \ldots, Z_{n^4}^{(n)}$ in such a manner that

$$\sup_i \operatorname{diam} Z_i^{(n)} = 0(n^{-2}). \tag{2}$$

(It may be obtained, for instance, by the division of the whole sphere by means of the geographical net).

It is possible to find positive integers $1 \leqslant p_n \leqslant m$, $1 \leqslant a_n \leqslant n^4$ such that, for some $t, t' \in Z_{a_n}^{(n)}$ and some $1 \leqslant i_n < j_n \leqslant (m+1)n^4$,

$$f_{p_n}(t) = s_{i_n}^{(n)},\ f_{p_n}(t') = s_{j_n}^{(n)}.$$

There follows, according to (1) and (2), the assertion of the Lemma.

Clearly, the Lemma holds true, if the numbers 2, 4 are replaced by arbitrary positive integers p, q with $p < q$.

Proof of Theorem 2′

For simplicity, only the case $n = 3$ will be considered.

The proof for $n > 3$ can be adduced without any essential changes.

Let Q be the boundary of a fixed 3-dimensional convex body with the centre of symmetry 0.

---

*) This Lemma may also be deduced from more general topological theorems.

**) The symbol $|s_i s_j|$ denotes the length of the segment $s_i s_j$; the symbol diam stands for diameter.

Let

(3) $$r = \inf_{A \in \Theta} |OA|, \qquad R = \sup_{A \in Q} |OA| .$$

1. Space Ξ and function $\Delta(t)$. Let Ξ denote the class of all convex ovals with centre of symmetry 0 which satisfy the condition

$$r \leqslant |OA| \leqslant R \quad \text{for} \quad A \in \Delta .$$

Let Δ and $\Delta' \in \Xi$ be given. Put Δ and Δ' in the *nearest position*, i. e. in such a position in a common plane, that the quantity

$$\sup_{\substack{A \in \Delta,\ A' \in \Delta' \\ |\sphericalangle AOA'| = 0}} \big||OA| - |OA'|\big| \,{}^{*)}$$

has a minimum.

This minimum will be called the *distance* between Δ and Δ' and denoted by the symbol $\varrho(\Delta, \Delta')$.

1.1. Consider sphere S' with radius 1 and centre 0. Denote by $\Delta(t)$ the intersection of Q with a plane perpendicular to direction $0t$.

1.2. *There exists a constant $M_1 > 0$ such that*

$$\varrho\big(\Delta(t), \Delta(t')\big) \leqslant M_1 \cdot |\sphericalangle tOt'|$$

for every $t, t' \in S$.

In fact, it is easy to verify that, for $A, A' \in Q$,

$$\big||OA| - |OA'|\big| \leqslant \sqrt{1 + \left(\frac{R}{r}\right)^2} \cdot |\sphericalangle AOA'| .$$

2. Space Ω, functions $h_i(\Delta)$ and $f_i(t)$. Let

$$\Omega = \{\Delta \in \Xi : \Delta \text{ is an octagon}\} .$$

Ω is a metric space with the metric ϱ.

2.1. Let $\Delta \in \Omega$. Enumerate the vertices of Δ according to one of the two orientations of the plane and denote them by the letters $A_1, A_2, \ldots, A_8$. Let $A_i = (x_i, y_i)$. Let the point $(\mu_1, \mu_2, \mu_3, \mu_4)$ where:

(4) $$\mu_1 = \frac{\begin{vmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}, \quad \mu_2 = \frac{\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}, \quad \mu_3 = \frac{\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}, \quad \mu_4 = \frac{\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_5 & y_5 & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}$$

of the Euclidean space E^4 correspond to Δ

*) The symbol $|\sphericalangle AOA'|$ denotes the measure of the angle $\sphericalangle AOA'$.

Since the vertices of Δ may be enumerated in a number of different ways, we obtain 8 points of E^4:

$$h_i(\Delta) = (\mu_1^{(i)}, \mu_2^{(i)}, \mu_3^{(i)}, \mu_4^{(i)}) \quad (i = 1, 2, \ldots, 8). \tag{5}$$

2.2. *The numbers* $\mu_i^{(j)}$ $(i = 1, 2, 3, 4;\ j = 1, 2, \ldots, 8)$ *are affine invariants of* Δ.

This follows from the fact that the formulae (4) express the quotients of areas of triangles having a common base.

2.3. *Let* δ *be an arbitrary real number* >0. *The functions* $h_1(\Delta), h_2(\Delta), \ldots, h_8(\Delta)$ *satisfy the Lipschitz conditions as the functions of the co-ordinates* $x_1, y_1, \ldots, x_8, y_8$ *of vertices of* Δ *on the set*

$$\left\{\Delta \in \Omega\colon \sup_{1 \leqslant i \leqslant 4} \begin{vmatrix} x_i & y_i & 1 \\ x_{i+1} & y_{i+1} & 1 \\ x_{i+2} & y_{i+2} & 1 \end{vmatrix}^2 \geqslant \delta \right\}.$$

2.4. Let $T = \Delta^{-1}(\Omega)$ (i. e. $T = \{t \in S\colon \Delta(t)$ is an octagon$\}$). Define on T the functions:

$$f_i(t) = h_i(\Delta(t)) \quad (i = 1, 2, \ldots, 8).$$

The values of these functions belong to E^4.

3. Two lemmata concerning space Ω and functions $h_i(\Delta)$. Let $\Delta \in \Omega$. Denote by $\tilde{\Delta}$ the class of Δ' $(\Delta' \in \Omega)$ affinely equivalent to Δ, i. e.

$$\tilde{\Delta} = h_1^{-1}(h_1(\Delta)).$$

3.1. *For every* $\Delta_0 \in \Omega$ *there exists* $\varepsilon_{\Delta_0} > 0$ *such that each of the functions* $h_1(\Delta), h_2(\Delta), \ldots, h_8(\Delta)$ *satisfies the Lipschitz condition in* ε_{Δ_0}-*neighbourhood of the set* $\tilde{\Delta}_0$.

This property is a consequence of 2.3 and of the following properties:

3.11. Let $\Delta_0 \in \Omega$. There are numbers $\varepsilon'_{\Delta_0} > 0$, $M_{\Delta_0} > 0$ such that, if $\Delta, \Delta' \in \Omega$, $\varrho(\Delta, \Delta_0) < \varepsilon'_{\Delta_0}$, $\varrho(\Delta', \Delta_0) < \varepsilon'_{\Delta_0}$; then, putting Δ and Δ' in the nearest position (see 1) it is possible to establish a correspondence: $A_i \leftrightarrow A'_i$ between the vertices of Δ and Δ', such that

$$|A_i A'_i| \leqslant M_{\Delta_0} \cdot \varrho(\Delta, \Delta') \quad (i = 1, 2, \ldots, 8).$$

3.12. If $\Delta, \Delta' \in \Omega$ and U is an affine transformation such that $U(\Delta)$ and $U(\Delta')$ belong to Ω, then $\varrho(U(\Delta), U(\Delta')) \leqslant \frac{R}{r} \cdot \varrho(\Delta, \Delta')$.

3.2. *For every* $\Delta_0 \in \Omega$, $\varepsilon > 0$ *there is* $\eta > 0$ *such that, if*

$$|(\mu_1, \mu_2, \mu_3, \mu_4)\, h_1(\Delta_0)| < \eta$$

there then exists an element $\Delta \in \Omega$ *with* $\varrho(\Delta, \Delta_0) < \varepsilon$, $h_1(\Delta) = (\mu_1, \mu_2, \mu_3, \mu_4)$.

4. Let us return to the proof of Theorem 2′. Suppose Q is universal for all centrally symmetric octagons. Let Δ_0 be a regular octagon (then $h_1(\Delta_0) = h_2(\Delta_0) = \ldots = h_8(\Delta_0)$).

On the basis of 3.2 it is possible to find a four-dimensional cube K with centre $h_1(\Delta_0)$, such that

$$K \subset h_1(\{\Delta \in \Omega\colon\ \varrho(\Delta, \widetilde{\Delta}_0) < \varepsilon_{\Delta_0}\})\ ^{*)}. \tag{6}$$

According to 3.1 and 1.2, all the functions $f_1(t), f_2(t), \ldots, f_8(t)$ satisfy the Lipschitz condition on the set

$$Z = f_1^{-1}(K) \cup f_2^{-1}(K) \cup \ldots \cup f_8^{-1}(K).$$

This contradicts the Lemma, because, according to (6), $f_1(Z) \cup f_2(Z) \cup \cup \ldots \cup f_8(Z) = K$.

Problems

1. To give an example of separable B-space which is universal for all finite dimensional B-spaces, but which is not universal for all separable B-spaces.

2. Suppose the separable B-space E is universal for all two-dimensional B-spaces. Is E universal for all finite dimensional ones?

3. To show a possibly small finite system of centrally symmetric ovals, for which no centrally symmetric convex body is universal.

It is easy to prove on the basis of Theorem 2′ that there exists such a system.

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*) The symbol $\varrho(\Delta, \widetilde{\Delta}_0)$ denotes here the distance between a point and a set.

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MATHEMATICS

On Homomorphisms not Induced by Mappings

by

T. TRACZYK

Presented by A. MOSTOWSKI on December 12, 1957

Let $\boldsymbol{A}$ be a σ-complete Boolean algebra and let $\mathfrak{S}$ and $\boldsymbol{S}$ denote respectively the set of all prime ideals of $\boldsymbol{A}$ and the Stone field of subsets of $\mathfrak{S}$ isomorphic to $\boldsymbol{A}$.

Let $\boldsymbol{N}$ be the σ-ideal of subsets of $\mathfrak{S}$ generated by the class of all sets of the form

$$\mathfrak{s}(A) - \sum_{k=1}^{\infty} \mathfrak{s}(A_k), \tag{1}$$

where $A, A_k \in \boldsymbol{A}$, $A = \sum_{k=1}^{\infty} A_k$ and $\mathfrak{s}$ is the Stone isomorphism *) of $\boldsymbol{A}$ on $\boldsymbol{S}$. The ideal $\boldsymbol{N}$ is the class of all subsets of sets $\sum_{k=i}^{\infty} N_k$, where N_k is of the form (1).

Let $\boldsymbol{Z}$ denote the class of sets $Z \subset \mathfrak{S}$ of the form

$$Z = S - N_1 + N_2,$$

where $S \in \boldsymbol{S}$ and $N_1, N_2 \in \boldsymbol{N}$.

Then (see [1], p. 245):

(i) *$\boldsymbol{Z}$ is a σ-field of subsets of $\mathfrak{S}$.*

(ii) *The mapping $\bar{\mathfrak{s}}$ defined by the formula*

$$\bar{\mathfrak{s}}(A) = [\mathfrak{s}(A)]\ {}^{**)} \qquad \textit{for} \qquad A \in \boldsymbol{A}$$

is an isomorphism of $\boldsymbol{A}$ on the σ-quotient algebra $\boldsymbol{Z}/\boldsymbol{N}$.

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*) $\mathfrak{s}(A)$ denotes the set of all prime ideals in $\boldsymbol{A}$ which do not contain A, $A \in \boldsymbol{A}$.

**) By $[X]$ we denote the element of a quotient algebra $\boldsymbol{X}/\boldsymbol{I}$ determined by X, i. e. the class of all sets $Y \in \boldsymbol{X}$ for which $X - Y + Y - X \in \boldsymbol{I}$.

Let A_1 be another σ-complete Boolean algebra and let $\mathfrak{S}_1, \boldsymbol{S}_1, \boldsymbol{N}_1, \boldsymbol{Z}_1, \mathfrak{s}_1, \bar{\mathfrak{s}}_1$ have analogous meanings. Then (see [2], p. 246):

(iii) *Every σ-homomorphism f of $\boldsymbol{Z}/\boldsymbol{N}$ in $\boldsymbol{Z}_1/\boldsymbol{N}_1$ is induced by a mapping φ of $\mathfrak{S}_1$ in $\mathfrak{S}$, i. e.:*

$$\varphi^{-1}(Z) \in \boldsymbol{Z} \quad \textit{and} \quad f([Z]) = [\varphi^{-1}(Z)] \quad \textit{for every} \quad Z \in \boldsymbol{Z}\,.$$

Now let $\boldsymbol{X}$ be a σ-field of subsets of a set $\mathfrak{X}$ and $\boldsymbol{I}$ and σ-ideal in $\boldsymbol{X}$.

Professor R. Sikorski has posed a question ([3], Problem 99), whether every σ-homomorphism of $\boldsymbol{Z}/\boldsymbol{N}$ in $\boldsymbol{X}/\boldsymbol{I}$ is induced by a mapping. The negative answer is given by the following theorem:

(iv) *Suppose that $A_1 = \boldsymbol{X}/\boldsymbol{I}$, i. e. that algebra $\boldsymbol{X}/\boldsymbol{I}$ is isomorphic to $\boldsymbol{Z}_1/\boldsymbol{N}_1$. Then the following conditions are equivalent:*

(1) *isomorphism $\bar{\mathfrak{s}}_1^{-1}$ of $\boldsymbol{Z}_1/\boldsymbol{N}_1$ onto $\boldsymbol{X}/\boldsymbol{I}$ is induced by a mapping;*

(2) *every σ-homomorphism h of any algebra $\boldsymbol{Z}/\boldsymbol{N}$ into $\boldsymbol{X}/\boldsymbol{I}$ is induced by a mapping;*

(3) *there exists a homomorphism n of $\boldsymbol{X}/\boldsymbol{I}$ in $\boldsymbol{X}$ such that $n([X]) \in \boldsymbol{X}$, i. e. $n([X]) = [X]$.*

Proof of implication $(1) \rightarrow (3)$:

Suppose that (1) is satisfied. Then there exists a mapping φ of $\mathfrak{X}$ in $\mathfrak{S}_1$, which induces an isomorphism $\bar{\mathfrak{s}}_1^{-1}$ of $\boldsymbol{Z}_1/\boldsymbol{N}_1$ in $\boldsymbol{X}/\boldsymbol{I}$, i. e.:

$$\varphi^{-1}(Z) \in \boldsymbol{X} \quad \text{and} \quad \bar{\mathfrak{s}}_1^{-1}([Z]) = [\varphi^{-1}(Z)] \quad \text{for} \quad Z \in \boldsymbol{Z}_1\,.$$

The formula $n([X]) = \varphi^{-1}\mathfrak{s}_1([X])$ defines a homomorphism of $\boldsymbol{X}/\boldsymbol{I}$ into $\boldsymbol{X}$ such that $n([X]) \in [X]$.

Indeed:

$$[\varphi^{-1}\mathfrak{s}_1([X])] = \bar{\mathfrak{s}}_1^{-1}\big([\mathfrak{s}_1([X])]\big) = \bar{\mathfrak{s}}_1^{-1}\bar{\mathfrak{s}}_1([X]) = [X]\,.$$

The proof of implication $(3) \rightarrow (2)$.

It follows from the identities

$$\Big[n\Big(\sum_{k=1}^{\infty}[X_k]\Big)\Big] = \sum_{k=1}^{\infty}[X_k] = \sum_{k=1}^{\infty}[n([X_k])] = \Big[\sum_{k=1}^{\infty}n([X_k])\Big]$$

that

$$n\Big(\sum_{k=1}^{\infty}[X_k]\Big) - \sum_{k=1}^{\infty}n([X_k]) \in \boldsymbol{I}\,. \tag{a}$$

Let then h be an arbitrary σ-homomorphism of $\boldsymbol{Z}/\boldsymbol{N}$ into $\boldsymbol{X}/\boldsymbol{I}$, and let $g(S) = nh([S])$ for $S \in \boldsymbol{S}$.

This formula defines a homomorphism g of $\boldsymbol{S}$ into $\boldsymbol{X}$. There exists ([1], p. 10) a mapping ψ of $\mathfrak{X}$ into $\mathfrak{S}$ which induces the homomorphism g, i. e.:

$$\psi^{-1}(S) \in \boldsymbol{X} \quad \text{and} \quad g(S) = \psi^{-1}(S) \quad \text{for every} \quad S \in \boldsymbol{S}\,.$$

If $N = \mathfrak{s}(A) - \sum_{k=1}^{\infty} \mathfrak{s}(A_k)$, where $A = \sum_{k=1}^{\infty} A_k$, then

$$\psi^{-1}(N) = \psi^{-1}\mathfrak{s}(A) - \sum_{k=1}^{\infty} \psi^{-1}\mathfrak{s}(A_k) = g\mathfrak{s}(A) - \sum_{k=1}^{\infty} g\mathfrak{s}(A_k) =$$

$$= nh\big([\mathfrak{s}(A)]\big) - \sum_{k=1}^{\infty} nh\big([\mathfrak{s}(A_k)]\big) = nh\bar{\mathfrak{s}}(A) - \sum_{k=1}^{\infty} nh\bar{\mathfrak{s}}(A_k).$$

Since $h\bar{\mathfrak{s}}$ is a σ-homomorphism and $A = \sum_{k=1}^{\infty} A_k$, it follows that

$$nh\bar{\mathfrak{s}}(A) = n\Big(\sum_{k=1}^{\infty} h\bar{\mathfrak{s}}(A_k)\Big).$$

Hence, we have:

$$\psi^{-1}(N) = n\Big(\sum_{k=1}^{\infty} h\bar{\mathfrak{s}}(A_k)\Big) - \sum_{k=1}^{\infty} nh\bar{\mathfrak{s}}(A_k) =$$

$$= n\Big(\sum_{k=1}^{\infty} [X_k]\Big) - \sum_{k=1}^{\infty} n([X_k]) \in \boldsymbol{I}$$

on account of (a), where $[X_k] = h\bar{\mathfrak{s}}(A_k)$.

This fact implies that $\psi^{-1}(N) \in \boldsymbol{I}$ for every $N \in \boldsymbol{N}$.

If $Z \in \boldsymbol{Z}$, then $Z = S - N_1 + N_2$, where $S \in \boldsymbol{S}$, $N_1, N_2 \in \boldsymbol{N}$ and, consequently, $\psi^{-1}(Z) \in \boldsymbol{X}$.

Thus, ψ induces ([4], p. 19) a σ-homomorphism of $\boldsymbol{Z}/\boldsymbol{N}$ in $\boldsymbol{X}/\boldsymbol{I}$. Let us denote it by $\bar{g}$. By definition

(b) $$\bar{g}([Z]) = [\psi^{-1}(Z)] \quad \text{for every} \quad Z \in \boldsymbol{Z}.$$

We shall prove that g is identical with h, i. e. that ψ induces h. In fact,

(c) $$\bar{g}([S]) = [\psi^{-1}(S)] = \big[nh([S])\big] = h([S]) \quad \text{for every} \quad S \in \boldsymbol{S}.$$

If Z is an arbitrary set belonging to the field $\boldsymbol{Z}$, then there is a set $S \in \boldsymbol{S}$ such that $[Z] = [S]$.

Therefore

$$h([Z]) = h([S]) = \bar{g}([S]) = \bar{g}([Z]) = [\psi^{-1}(Z)]$$

on account of (b) and (c).

The implication (2) $\Rightarrow$ (3) is trivial.

Condition (3) is not always satisfied. This results from the following theorem of J. v. Neumann and Stone [1].

(v) *Let* $\boldsymbol{X}$ *be a field of subsets of a space* $\mathfrak{X}$ *containing the ideal* $\boldsymbol{I}$ *of all subsets* A *of* $\mathfrak{X}$ *with* $\overline{\overline{A}} < \aleph_\beta$. *Suppose that*

(a_1) $\overline{\overline{\boldsymbol{X}}} = \aleph_\alpha > \aleph_\beta$,

(a_2) *there exists a class* $\boldsymbol{M} \subset \boldsymbol{X}$ *of disjoint sets and a class* $\boldsymbol{N} \subset \boldsymbol{X}$ *of disjoint sets such that* $\overline{\overline{\boldsymbol{M}}} \geqslant \aleph_\beta$, $\overline{\overline{\boldsymbol{N}}} = \aleph_\alpha$ *and* $\overline{\overline{AB}} = 1$ *for every* $A \in \boldsymbol{M}$ *and every* $B \in \boldsymbol{N}$.

Then there is no homomorphism n *of* $\boldsymbol{X}/\boldsymbol{I}$ *into* $\boldsymbol{X}$ *such that* $n([X]) \in [X]$.

Let $\aleph_\alpha = 2^{2^{\aleph_0}}$ and $\aleph_\beta = \aleph_1$. Let $\mathfrak{Y}$ be a set of cardinal number $2^{2^{\aleph_0}}$ and let us denote by $\boldsymbol{I}$ the class of all enumerable sets of the Cartesian product $\mathfrak{X} = \mathfrak{Y} \times \mathfrak{Y}$. Let ξ be a fixed element of $\mathfrak{Y}$. By A_ξ we mean a set of points (ξ, η), $\eta \in Y$ and by M the class of all sets A_ξ where $\xi \in \mathfrak{Y}$.

Similarly, for a fixed η let B_η be a set of the points (ξ, η), $\xi \in Y$ and $\boldsymbol{N}$ the class of all B_η, where $\eta \in \mathfrak{Y}$. The cardinal numbers of $\boldsymbol{M}$ and $\boldsymbol{N}$ are $2^{2^{\aleph_0}}$.

Now let $\boldsymbol{X}$ be the smallest σ-field of subsets of $\mathfrak{Y} \times \mathfrak{Y}$ containing the elements of classes $\boldsymbol{M}, \boldsymbol{N}$ and $\boldsymbol{I}$. It is easy to see that the cardinal number of X is $2^{2^{\aleph_0}}$.

The other conditions of the theorem of J. v. Neumann and Stone are also satisfied. Of course I is a σ-ideal in X.

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EXPERIMENTAL PHYSICS

Dependence of Polarization of Fluorescence on the Concentration of Luminescent Molecules in Plexiglassphosphors

by

M. CZAJKOWSKI and J. GRZYWACZ

Presented by A. JABŁOŃSKI on December 7, 1957

The authors investigated the dependence of the degree of polarization of fluorescence light on the concentration of naphtacene and eosin in rigid solutions, using organophosphors obtained by dissolving the dye at appropriate concentrations in methyl methacrylate monomer. The monomer samples were placed within test-tubes which were sealed up in order to avoid variations in the concentration. Polymerization proceeded at a very slow rate (requiring about 20 days) at a temperature of 30° C. The coefficient of contraction k ($k = 1.36$) was calculated in order to introduce a correction in the concentration of the dye in the polymer. The luminophors obtained by this method exhibited birefringence which was eliminated by heating to about 150° C. and cooling at an exceedingly slow rate.

The degree of polarization was measured with an Arago compensator and Savart plate, as shown in Fig. 1. Plane polarized light was used for excitation; the light, which was obtained from a LKW type mercury lamp, was made to pass through a double Wood filter.

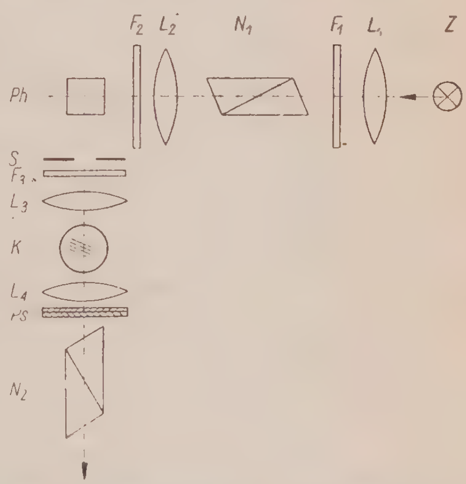

Fig. 1. Experimental arrangement for determination of the degree of polarization. Z — light source, L_1-L_4 — lenses, F_1-F_3 — filters, N_1-N_2 — Nicols, Ph — phosphor, S — slit, K — Arago compensator, P_s — Savart plate

The degree of polarization was calculated from the Kawski-Neumann formula [1] for four plate Arago compensator:

$$(1)\qquad P = \frac{\left(1-\frac{1}{n^2}\right)\sin^2\alpha}{\left(1+\frac{1}{n^2}\right)\left[2-\left(1+\frac{1}{n^2}\right)\sin^2\alpha\right]-\frac{3\cos\alpha}{n^2}\sqrt{n^2-\sin^2\alpha}};$$

here, n denotes the refractive index of the plates, and α — the angle at which the fringes disappear.

In presenting the results it will be more convenient to employ the emission anisotropy r as introduced by A. Jabłoński [2], [3] instead of the degree of polarization P; for plane polarized exciting light, r is given by:

$$(2)\qquad r = \frac{I_{\parallel}-I_{\perp}}{I_{\parallel}+2I_{\perp}} = \frac{2P}{3-P},$$

where $I_{\parallel}$ denotes the component of the emitted intensity parallel to the electric vector of the exciting light,

$I_{\perp}$ — the component of that perpendicular to the electric vector of the exciting light, and

P — the degree of polarization.

It should be noted that, since, in the case of isotropic solution, $-1/3 \leqslant P \leqslant 1/2$ [4], [5], the values of r are limited to $-0.2 \leqslant r \leqslant 0.4$. The results of the measurements are shown in Table I and plotted in

TABLE I

Naphthacene

P (%)	13.7	16.5	19.8	22.2	25.7
c	$1.36\cdot10^{-5}$	$2.1\cdot10^{-5}$	$4.0\cdot10^{-5}$	$7\cdot10^{-5}$	$1.36\cdot10^{-4}$
r_d	0.095	0.116	0.141	0.160	0.187
r_t	0.090	0.111	0.140	0.160	0.180
$1/r_d$	10.5	8.6	7.09	6.20	5.31
$1/r_t$	11.1	9.0	7.10	6.20	5.60
$1/c$	$7.35\cdot10^{4}$	$4.76\cdot10^{4}$	$2.5\cdot10^{4}$	$1.42\cdot10^{4}$	$7.35\cdot10^{3}$

Eosin

P (%)	22.2	29.7	33.4	34	37
c	$5.4\cdot10^{-6}$	$1.36\cdot10^{-5}$	$2.72\cdot10^{-5}$	$4.1\cdot10^{-5}$	$5.4\cdot10^{-5}$
r_d	0.159	0.219	0.250	0.255	0.281
r_t	0.149	0.219	0.268	0.275	0.296
$1/r_d$	6.30	4.54	4.00	3.90	3.55
$1/r_t$	6.90	4.54	3.9	3.60	3.42
$1/c$	$1.85\cdot10^{5}$	$7.35\cdot10^{4}$	$3.67\cdot10^{4}$	$2.43\cdot10^{4}$	$1.85\cdot10^{4}$

P — rate of polarization
c — concentration
r_d — experimental value of emission anisotropy
r_t — theoretical values of emission anisotropy calculated by means of Eq. (4)

Figs. 2 and 3 (the abscissae represent the reciprocals of the concentrations, the ordinates the reciprocals of the emission anisotropy).

It will be seen from the diagrams that r grows as the concentration of the dye rises; this is a somewhat surprising result, as, in general, observational data [9], [10] and theoretical considerations [3], [6] point to a decrease of polarization with the concentration.

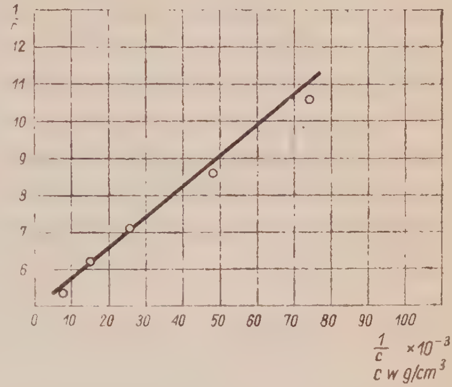

Fig. 2. $1/r$ versus $1/c$ for naphthacene (the abscissae $1/c \times 10^{-3}$, – concentration c in g per cm³.). ∘ – experimental points, straight line – theoretical curve drawn according to Eq. (4)

It should be stressed that the short-wave component of the light used for excitation ($\lambda = 3120.66$ Å and $\lambda = 3650.15$ Å Hg) is partly absorbed by plexiglass. The molecules of the dye absorb rather the longer wavelengths (from $\lambda = 3650.15$ Å upwards).

It seems possible to explain the rise in the emission anisotropy with higher molecular concentrations of the dye in the luminophor as follows: Kallmann et al. [7], [8] found that in a number of cases the exciting energy absorbed by the solvent is transmitted to the luminescent molecules, which are thus indirectly excited (*via* the solvent). It may be assumed that the luminescent molecules excited in this way emit unpolarized fluorescence light. In the case under consideration, fluorescence light would be emitted both by molecules indirectly excited *via* the molecules of the solvent and by molecules excited directly by the exciting light. The latter would emit polarized fluorescence. It may be assumed that, in the first approximation, the intensity of the indirectly excited fluorrescence is independent of the concentration (the total energy absorbed by the plexiglass is independent of the concentration of the dye, and

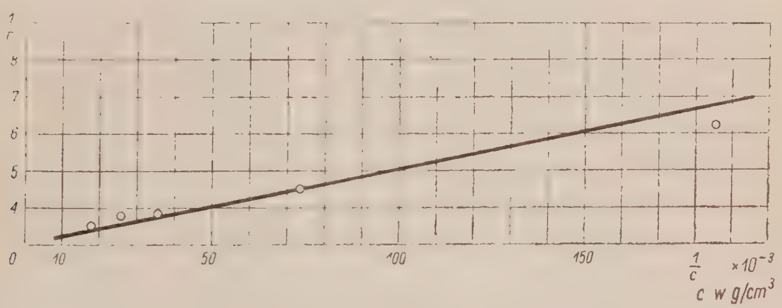

Fig. 3. $1/r$ versus $1/c$ for eosin (the abscissae $1/c \times 10^{-3}$, concentration c in g per cm³).

thus may be provisionally assumed to be entirely used up in exciting the molecules of the dye); on the other hand, the intensity of the directly excited fluorescence is proportional to the concentration of the dye. Hence, it may be concluded that the emission anisotropy grows as the concentration rises (provided the self-depolarization arising from migration of energy among the molecules of the dye may be neglected).

We shall now try to give a quantitative account of the way r depends on c. Denoting the number of molecules of the dye excited directly by N_d, and the number of those excited indirectly by N_i, and assuming $r = r_0$ for $P = P_0$, the following simple expression for r is obtained:

$$r = r_0 \frac{N_d}{N_d + N_i}. \tag{3}$$

In the first approximation, N_d may be assumed to be a linear function of the concentration, whereas N_i is almost independent of the latter (N_i is assumed constant):

$$N_d = Ac, \qquad N_i = B = \text{const}.$$

Hence, we have

$$r = r_0 \frac{Ac}{Ac + B} = r_0 \frac{1}{1 + \beta/c}, \tag{4'}$$

or

$$\frac{1}{r} = \frac{1}{r_0}(1 + \beta/c), \tag{4}$$

with

$$\beta = \frac{B}{A} = \text{const}.$$

Thus, in (4) we have two constants which may be fitted to experimental data. In accordance with various authors [9], [10] the following values of the fundamental emission anisotropy r_0 have been assumed: $r_0 = 0.2$ for naphtacene, and $r_0 = 0.344$ for eosin. The following values of β have been assumed: $\beta = 1.68 \cdot 10^{-5}$ g./cm.^{3} for naphtacene, and $\beta = 7.3 \cdot 10^{-6}$ g./cm.^{3} for eosin. With these values, expression (4) is found to be in satisfactory agreement with the results of measurements. Table I and the curves in Figs. 2 and 3 serve to compare the latter with the calculated values. The fact that the concentrations employed in the present investigation are restricted to a relatively narrow range is a consequence of the low solubility of naphtacene and, more particularly, eosin in methyl methacrylate monomer, which makes it impossible to obtain luminophors of higher concentrations.

The authors wish to express their very cordial thanks to Professor A. Jabłoński for suggesting interpretations and for his numerous valuable hints in the course of the present investigation.

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EXPERIMENTAL PHYSICS

On the Exciting Wave Length Dependence of the Ratio of the Yields of Phosphorescence and Fluorescence

by

R. BAUER and A. BĄCZYŃSKI

Presented by A. JABŁOŃSKI on December 9, 1957

The authors investigated the ratio of the phosphorescence and fluorescence yields as to its dependence on the wave length of the exciting light. The investigation was carried out with a 10^{-4} g./g. solution of fluoresceine in boric acid. The luminophor was excited with light of a spectral width of 30 Å.

Kantargian [1] investigated the temperature dependence of the ratio of the slow to rapid fluorescence intensities. He showed that the ratio of the $F \to M$ transition and the $F \to N$ transition probabilities is temperature-independent. The states N, F, M are states appearing in Jabłoński's level diagram [2]. The aim of the present work is to investigate, whether the $F \to M$ transition probability depends on the wave length of the exciting light.

The measurements were carried out with a photoelectronic device (Fig. 1) consisting of light source, Zeiss mirror monochromator, phosphoro-

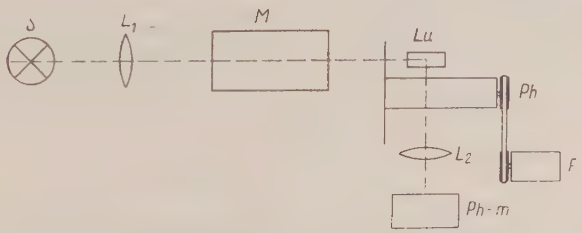

Fig. 1. Block diagram of the experimental arrangement. S — source of light, L_1, L_2 — lenses, M — monochromator, Lu — luminophor, Ph — phosphoroscope, F — electric motor, $Ph\text{-}m$ — photomultiplier

scope, 1P-21 photomultiplier, RFT 1 KO-712 type cathode ray oscillograph, and valve voltmeter. A special phosphoroscope was constructed

for measuring the luminescence intensity in a direction perpendicular to that of the incidence of the exciting light. The principle of the phosphorocope is similar to that of the Becquerel phosphoroscope; however, it contains only one disk with slots. Instead of the second disk, a cylinder of considerable diameter, having two narrow, oblong slots opposite one another, was employed (Fig. 2). During one full turn, the monochromator and the photomultiplier "glance" twice at the luminophor, the former through the slots in the disk, and the latter through those in the cylinder; the luminophor is placed along the cylinder, which may be set with respect to the disk in such a manner as to let the exciting light fall on the luminophor and have the luminescence light fall simultaneously on the photomultiplier, the luminescence observed thus being in this case a mixture of fluorescence and phosphorescence. The cylinder may also be turned with respect to the disk to a position in which the luminescence light falls on the photomultiplier when the exciting light is shut off from the luminophor, so that only the phosphorescence is observed. The oscillograph acts as a broad-band amplifier for transmitting the amplified signal from the photomultiplier to the valve voltmeter.

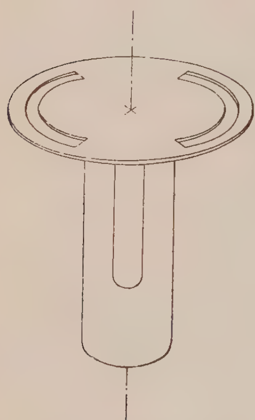

Fig. 2. Disk of phosphoroscope with cylinder

The luminescence yield is given by the following formula:

$$\eta = AF/I_0(\lambda)\,\mu(\lambda)\,, \tag{1}$$

where F is the voltage indicated by the voltmeter,
$I_0(\lambda)$ — the intensity of the incident light,
$\mu(\lambda)$ — the absorption coefficient,
A — proportionality factor.

The ratio of the phosphorescence and fluorescence yields is [3]

$$\frac{\eta_{Ph}}{\eta_F} = \frac{F_{Ph}}{F'_F - F_{Ph}}\,\frac{1-\exp\left[-\frac{t}{\tau}\right]}{1-\exp\left[-\frac{t_1}{\tau}\right]}, \tag{2}$$

where F_{Ph} and F'_F denote the respective readings on the voltmeter. The intensity of the fluorescence cannot be measured directly, as it is, in our case, always mixed with phosphorescence. In order to obtain the fluorescence intensity, F_{Ph} should be subtracted from F'_F. Thus, we have

$$F_F = F'_F - F_{Ph}\,.$$

$$F^{(0)}_{Ph} = F_{Ph}\left(1-\exp\left[-\frac{t}{\tau}\right]\right)\left(1-\exp\left[-\frac{t_1}{\tau}\right]\right)^{-1} \tag{3}$$

represents the phosphorescence intensity in the case of a continuous illumination (cf. [3]). In the above formula

t_1 — denotes the duration of each illumination period,
t_2 — the corresponding dark period,
τ — the mean decay time of the luminescence, and
$t = t_1 + t_2$.

In Fig. 3 η_{Ph}/η_F is plotted versus the wave length λ_{in} of the exciting light. Linschitz showed that the ratio of the phosphorescence and fluo-

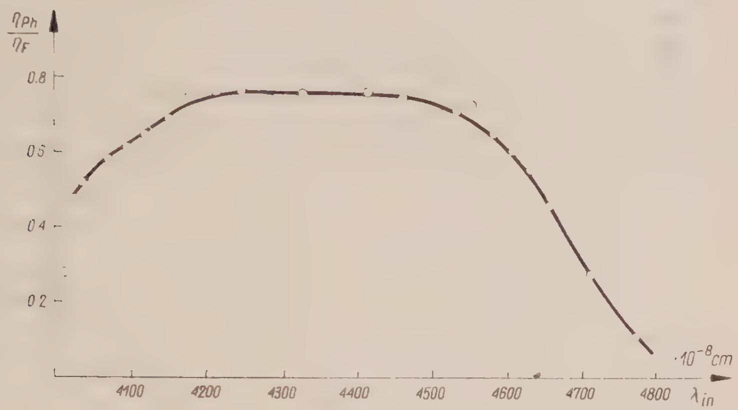

Fig. 3. Dependence of η_{Ph}/η_F on wavelength of exciting light

rescence intensities depends on the wave length of the exciting light. According to Pringsheim [4], he obtained a decrease of the ratio with increasing wave lengths λ_{in}, up to the long-wave limit of the absorption band where the ratio of the intensities becomes very small. The results of the present investigation are in accordance with those of Linschitz within the long-wave part of the region investigated.

The $M \to N$ transition probability is very low as compared with that of $M \to F$ transition at room temperatures, assuming it to be 0, we have (cf., e. g. [5]):

$$\frac{\eta_{Ph}}{\eta_F} = \frac{\gamma_{FM}}{\gamma_{FN} + W_F}, \tag{4}$$

where

γ_{FM} denotes the probability of the $F \to M$ transition,

γ_{FN} — the probability of $F \to N$ transition with emission of radiation and

W_F — the probability of $F \to N$ radiationless transition.

Up to $\lambda < 4550$ Å, the quantum fluorescence yield of fluorescein is practically independent of the wave length of the exciting light [6].

Thus, as Eq. (4) and inspection of Fig. 3 shows, for sufficiently short λ_{in}, the $F \to M$ transition probability depends on the wave length of the exciting light.

The dependence of η_{Ph}/η_F on λ_{in} may be explained on the basis of the Franck-Condon potential curves (Fig. 4). Within the long-wave

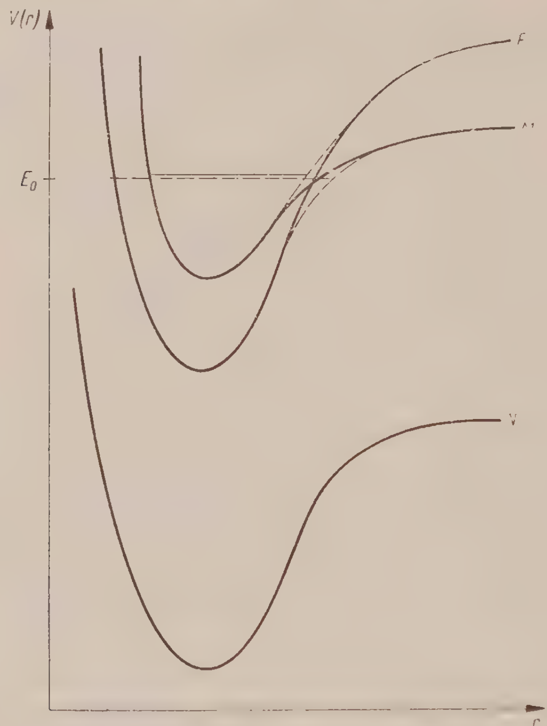

Fig. 4. Potential curves for a phosphorescent molecule

region, the probability of transitions from higher vibrational levels of state F to state M is higher than that from lower levels, and thus rises with decreasing λ_{in}, and reaches maximum when the vibrational level corresponding to the "intersection" of potential curves is excited. At still shorter $\lambda_{in}(\lambda_{in} < 4150$ Å), this probability decreases again.

According to the quantum-mechanics [7]

$$\gamma_{FM} \sim \left| \int \psi W \psi' d\tau \right|^2, \tag{5}$$

where ψ and ψ' are the eigenfunctions of the vibrational motion [in the states F and M respectively and W the part of perturbation energy depending on nuclear co-ordinates. Since W varies but slowly with nuclear co-ordinates, the integral is essentially the overlap-integral of the vibrational levels in which the molecule can come close to the intersection of two potential curves.

The authors express their thanks to Professor A. Jabłoński and Dr. D. Frąckowiak for discussion and numerous helpful hints received in the course of the present investigation.

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EXPERIMENTAL PHYSICS

Solution of the Schrödinger Equation for the Van der Waals Potential

by

A. BĄCZYŃSKI, W. BERDOWSKI and S. ŁĘGOWSKI

Presented by A. JABŁOŃSKI on December 9, 1957

In solving Schrödinger's equation for a number of problems, it is possible to obtain approximations only. The respective methods involve very tedious calculations, and, in some cases, do not even yield good approximations.

In 1953, Antonowicz [1]-[3] constructed a device for integrating Schrödinger's equation, in which advantage was taken of the analogy existing between the latter and that describing the motion of a magnetic needle in a magnetic field *). Recently this device has been improved in some respects; first of all, the replacement of the original cocoon suspension by a quartz thread should be mentioned. The former had the serious drawback of resulting in variations of the zero position of the magnetic needle; no such variations have been observed with the quartz suspension applied at present.

In order to make it possible for the device to be used for solving the Schrödinger equation, it must first be scaled. In earlier investigations [2] this was done by solving a known problem. Such a necessity no longer exists.

As already stated, the device is based on the analogy between the (one-dimensional, time-independent) Schrödinger equation

$$\frac{d^2\psi}{dr^2}+\frac{2\mu}{\hbar^2}[E_n-V(r)]\psi=0 \tag{1}$$

and the equation of motion of a magnetic needle in a magnetic field for small deflections of the needle

$$\frac{d^2a}{dt^2}+\frac{M}{I}[H_0-H(t)]a=0\,. \tag{2}$$

*) A description of the device will be found in [1]-[3].

Substitution of

$$x = \sqrt{\frac{2\mu}{\hbar^2}}\, r \tag{3}$$

into Eq. (1) yields

$$\frac{d^2\psi}{dx^2} + [E_n - V(x)]\psi = 0\,. \tag{4}$$

$H(t)$ and H_0 in Eq. (2) may be expressed as follows:

$$H(t) = A[V(t) - E'] + B\,, \tag{5}$$

$$H_0 = C_n + D\,, \tag{6}$$

where A and B denote constants of the device dependent on the coefficient of amplification of the amplifier, on the characteristic of the phototube and the intensity of the light incident on it, as well as on the geometrical dimensions of the Helmholtz coils:

C_n denotes a constant proportional to the current producing the constant field H_0, chosen so as to obtain the consecutive eigenfunctions,

E' is a constant depending on the system within which the potential curve has been traced, and

D is the restoring torque of the quartz thread.

By appropriately choosing the independent variable and substituting $H(t)$ and H_0 from Eqs. (5) and (6) in Eq. (2), we obtain

$$\frac{d^2\alpha}{d\tau^2} + \left[\frac{C_n + D - B}{A} + E' - V(\tau)\right]\alpha = 0\,, \tag{7}$$

where $\tau = \sqrt{\frac{MA}{I}}\, t$.

By comparing the latter equation with Eq. (4), the following expression is obtained for the energy eigenvalue:

$$E_n = \frac{C_n + D - B}{A} + E'\,. \tag{8}$$

Finally, Eq. (7) may be written as follows:

$$\frac{d^2\alpha}{d\tau^2} + [E_n - V(\tau)]\alpha = 0\,. \tag{9}$$

In order to calculate the eigenvalue of the energy, it is necessary to determine B, D, E', C_n.

If $V = \text{const.}$, then Eq. (9) is that of harmonic motion with angular frequency

$$\omega = \sqrt{E_n - V}\,. \tag{10}$$

In this case the α versus τ graph (which is obtained on photosensitive paper) is such a sinusoid that the distance λ between two consecutive

points of equal phase is given by

$$\lambda = \Omega R \sqrt{\frac{I}{MA}}\, T = \Omega R \frac{2\pi \sqrt{\frac{I}{MA}}}{\sqrt{E_n - V}}, \tag{11}$$

where Ω is the angular velocity of the motor displacing the potential curve and simultaneously turning the drum carrying the photographic paper, whilst R denotes the radius of the drum.

By substituting (8) in Eq. (11), we obtain a relation between the constants of the device and the distance λ:

$$\frac{C_n + D - B}{A} + E' - V = \frac{4\pi^2 \Omega^2 R^2 I}{MA} \frac{1}{\lambda^2}. \tag{12}$$

If the magnetic needle executes torsional vibrations in the absence of the fields ($H_0 = 0$, $B = 0$, $H(t) = 0$), a sinusoid is obtained for which $\lambda = \lambda_0$ may be measured. If the constant field $H_0 = 0$ and the direction of $H(t)$ is changed to the opposite, the magnetic needle executes torsional vibrations, and the known constant function $V_1 = \text{const.}$ passes before the slit of the phototube. Since $H(t) = \text{const.}$, a sinusoid is obtained on the photosensitive paper, for which the distance $\lambda = \lambda_1$ may be measured. Proceeding in a similar manner, if another known constant function $V_2 = \text{const.}$ is made to pass before the slit of the phototube, $\lambda = \lambda_2$ is obtained. By finding the n-th eigenfunction for a constant value of the field H_0 and letting the magnetic needle execute torsional vibrations at this field strength value, whilst keeping the field $H(t)$ at zero, a sinusoid is obtained for which $\lambda = \lambda_n$ may be measured. Substituting λ_0; V_1, λ_1; V_2, λ_2; λ_n in Eq. (12), a system of equations is obtained, whence the constants of the device, and thus the energy eigenvalue,

$$E_n = V_1 + \frac{V_1 - V_2}{\frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2}} \left(\frac{1}{\lambda_0^2} - \frac{1}{\lambda_1^2} + \frac{1}{\lambda_n^2} \right) \tag{13}$$

may be determined.

The energy eigenvalue may also be determined by measuring the corresponding periods T_0, T_1, T_2, T_n instead of λ_0, λ_1, λ_2, λ_n.

The correspondence between the co-ordinates r serving to express the potential curve and the ξ co-ordinates wherein the eigenfunction is obtained was determined by flashes. Let r_1 and r_2 denote the abscissae of two arbitrary points on the potential curve. At the moments when the points pass in front of the slit, flashes are activated marking the points ξ_1 and ξ_2 on the abscissae of the photosensitive paper. Between r and ξ the following relation holds:

$$r = r_1 + \frac{r_1 - r_2}{\xi_1 - \xi_2} (\xi - \xi_1). \tag{14}$$

It is the aim of the present investigation to find the vibrational energy levels and the corresponding eigenfunctions for the van der Waals Hg-A potential. The potential $V(r)$ has been assumed in the form

$$V(r) = \frac{C_2}{r^8} - \frac{C_1}{r^6}. \tag{15}$$

The constant C_1 was theoretically computed by Kuhn [4]; the constant C_2 was calculated assuming the potential curve minimum at $r = 3$ Å and a r^{-8} type repulsion term. It should be stressed that recent results [5] point to the necessity of modifiyng the potential curve. Obviously, this would result in a modification of the eigenfunctions and eigenvalues of the vibrational energy. In the present paper the centrifugal potential has been disregarded; thus, the results are those for rotationless states.

The eigenfunctions ψ_n for $n = 1, 2, 3, 4$ obtained with the device on photographic paper have been normalized. They are shown in Fig. 1.

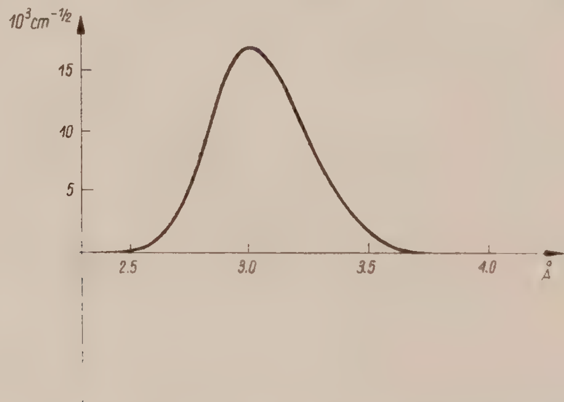

Fig. 1a. Normalized eigenfunctions obtained with device

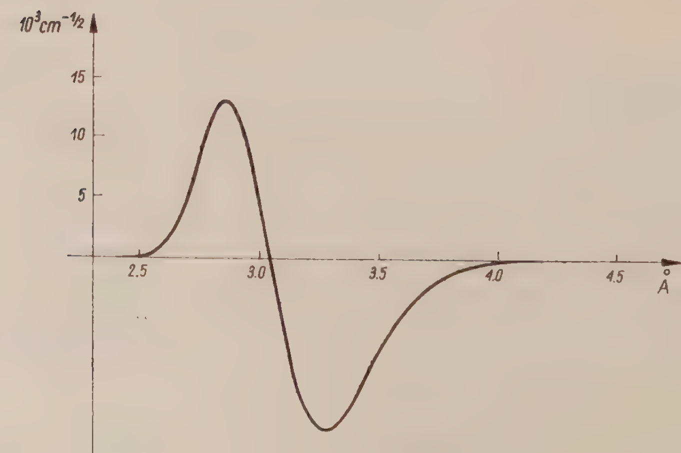

Fig. 1b

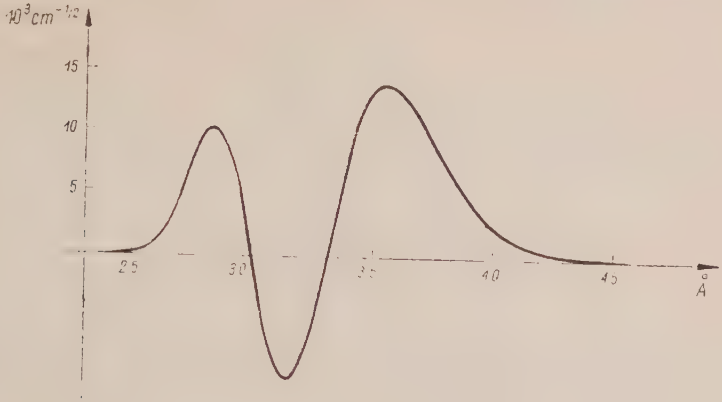

Fig. 1c

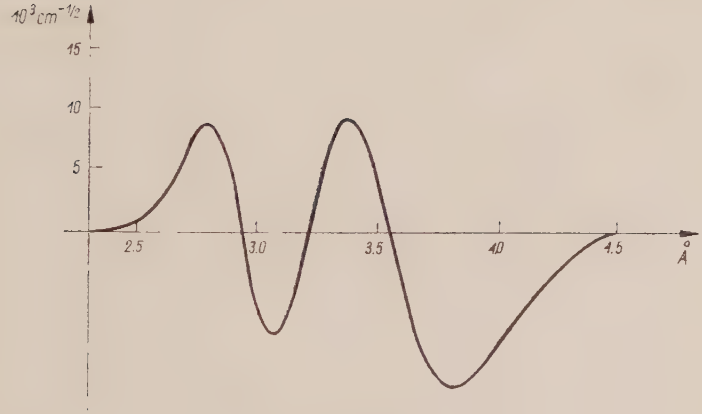

Fig. 1d

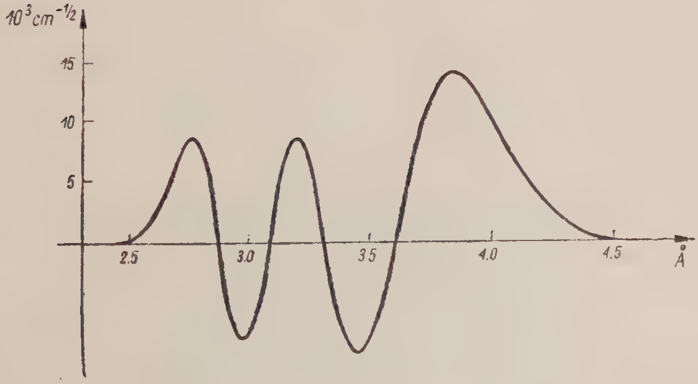

Fig. 1e

For some eigenfunction, orthonormality was checked (Table I).

TABLE I

$\psi_n\psi_m$	$\left\|\int\psi_n\psi_m dx\right\|$
$\psi_0\psi_1$	0.02130
$\psi_0\psi_2$	0.02110
$\psi_0\psi_3$	0.00125
$\psi_0\psi_4$	0.01895
$\psi_1\psi_4$	0.00850

The second eigenfunction as obtained with the device was compared with the second eigenfunction determined by the W. K. B. method and the second eigenfunction as calculated for an appropriately chosen Morse potential. The latter was chosen so as to achieve the best possible accordance with potential (15) in the neighbourhood of the minimum (Fig. 2).

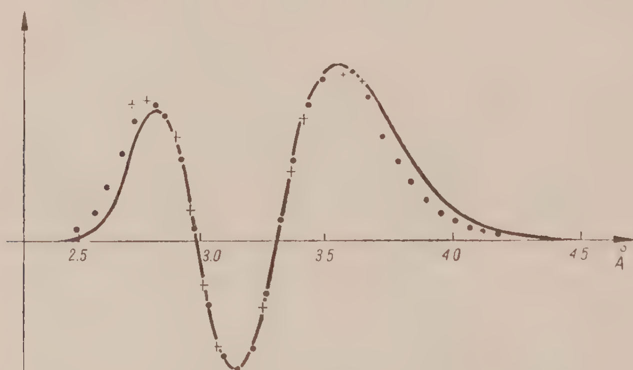

Fig. 2. — Eigenfunction obtained with device
× Eigenfunction calculated by W. K. B. method
• Eigenfunction for Morse potential

The eigenvalues corresponding to potential (15) as obtained from Eq. (13) were compared with those corresponding to the Morse potential already mentioned. The results are shown in Table II.

TABLE II

n	E_n (van der Waals)	E_n (Morse)
0	$-354.54\cdot10^{-16}$ erg	$-356.1\cdot10^{-16}$ erg
1	$-275.30\cdot10^{-16}$ „	$-299.7\cdot10^{-16}$ „
2	$-206.04\cdot10^{-16}$ „	$-248.2\cdot10^{-16}$ „
3	$-146.78\cdot10^{-16}$ „	$-201.5\cdot10^{-16}$ „
4	$-\ \ 94.84\cdot10^{-16}$ „	$-159.6\cdot10^{-16}$ „

The authors wish to express their very sincere thanks to Professor A. Jabłoński and Professor K. Antonowicz for the help received in the course of the present investigation.

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EXPERIMENTAL PHYSICS

Interaction Potential between a Mercury Atom and an Argon Atom

by

S. ŁĘGOWSKI

Presented by A. JABŁOŃSKI on December 9, 1957

The interaction forces between Hg- and A-atoms in a Van der Waals Hg-A molecule are calculated from experimental data for two electronic states: for the ground state and for one excited state. If a mercury atom in the excited state $^3P_1$ approaches an argon atom in its ground state $^1S_0$, the orientation of the angular momentum $J = 1$ of the Hg atom in the axial field of force of the atom pair produces two quantum states $\Omega = \pm 1$, and $\Omega = 0$. In view of the lack of sufficient experimental data suitable for the determination of the potential parameters for the excited state $\Omega = 0$, only potential curves for the ground and the excited state $\Omega = 1$ have been calculated.

In principle, the intermolecular potential between unlike molecules (in their ground state) could be determined experimentally from gas properties, but no measurements on gaseous mixtures of sufficient accuracy are so far available for the purpose. The difficulty lies partly in the fact that the properties of a gaseous mixture depend not only upon the forces between unlike molecules, but also strongly on the interaction between atoms of the same species, so that the dependence of a property on the potential between unlike molecules is partly masked by the dependence on the potentials between like molecules in the mixture. It is, however, possible to determine the potential between unlike molecules from the known interaction for like molecules.

For the interaction between atoms in non-polar molecules the Lennard-Jones [6]-[12] potential is used [1], viz.

$$V(r) = 4\varepsilon\left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right], \tag{1}$$

where $V(r)$ is the potential energy of two atoms at a distance r, ε the depth of the potential energy minimum, σ the value of r for which $V = 0$.

The position of the minimum is defined by

$$r_m = 2^{1/6}\sigma . \tag{2}$$

The empirical combining laws [1], [2] relating the force constants between unlike molecules to those between like molecules are

$$\sigma_{\alpha\beta} = \tfrac{1}{2}(\sigma_\alpha + \sigma_\beta) \tag{3}$$

and

$$\varepsilon_{\alpha\beta} = (\varepsilon_\alpha \varepsilon_\beta)^{1/2} , \tag{4}$$

where $\sigma_{\alpha\beta}$, $\varepsilon_{\alpha\beta}$ refer to the "unlike" interaction, and σ_α, ε_α; σ_β, ε_β; — to the "like" interaction. The first of these rules (3) clearly holds for rigid spherical molecules. The second rule (4) follows from an interpretation of the dispersion forces in terms of the polarizabilities of the individual molecules.

Potential parameters for the argon molecule A_2 were determined by many investigators [3]-[7] from viscosity and second virial coefficients. The results of their calculations are summarized in Table I.

TABLE I

ε_A/k (°K.)[a]	σ_A(Å)	Refs. for data
124	3.418	H. L. Johnston, E. R. Grilly [3]
116	3.465	M. Trautz, A. Melster, R. Zink, H. Binkele [4], [5]
119.8	3.405	A. Michels, Hub. Wijker, Hk. Wijker [6]
122	3.4	L. Holborn, J. Otto [7]
120.5	3.422	average value

a) — here k is the Boltzmann constant.

Force constants for mercury vapour were calculated by L. F. Epstein and M. D. Powers [8] from its viscosity. They obtained the following values:

$$\varepsilon_{Hg}/k = 851°\,K$$

and

$$\sigma_{Hg} = 2.898\ Å .$$

Substitution of these potential parameters into equations (3) and (4) leads to the following values for Hg+A:

$$\begin{aligned} \varepsilon_{HgA}/k &= 320.2°\,K ., \\ \sigma_{HgA} &= 3.16\ Å , \\ (r_{HgA})_m &= 3.539\ Å . \end{aligned} \tag{5}$$

In order to obtain force constants for an excited unlike molecule we must determine them from spectroscopic data, because no measurements of viscosity or second virial coefficient for excited molecules were performed.

The depth of the potential energy minimum for the excited Hg*-A molecule can be calculated from S. Mrozowski's [9]-[11] potential curves for a mercury molecule Hg_2 in the state $^3 1_u$ (one of the mercury atoms being in the excited state $^3P_1$ and the total molecule in the quantum state $\Omega = 1$). According to Mrozowski, the dissociation energy for this state is

$$\varepsilon_{\mathrm{Hg^*Hg}}/k = 1439^\circ\,\mathrm{K}\,. \tag{6}$$

From (4) and (6) we obtain

$$\varepsilon_{\mathrm{Hg}^*}/k = 1124.8^\circ\,\mathrm{K}\,. \tag{7}$$

The second parameter was determined from H. Kuhn's paper [12]. Kuhn has calculated the value of the following expression

$$4\varepsilon_{\mathrm{HgA}}(\sigma_{\mathrm{HgA}})^6 - 4\varepsilon_{\mathrm{Hg^*A}}(\sigma_{\mathrm{Hg^*A}}) = 5.961\cdot 10^{-59}\ \mathrm{erg\ cm^6}\,. \tag{8}$$

from the intensity distribution of the pressure broadened long wavelength wing of the mercury line 2537 Å.

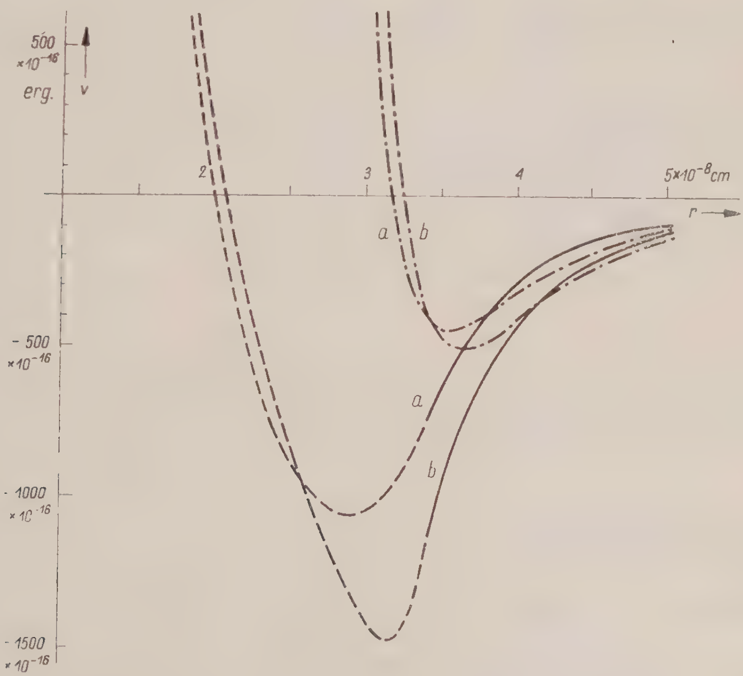

Fig. 1. Potential curves for a Hg-A molecule
(a) the ground state, (b) an excited state
——— quantitative; – – – – – qualitative Kuhn's curves;
–·–·–·– curves determined by the author

The force constants obtained by the present method for the excited molecule Hg*-A ($\Omega = 1$) are

$$\varepsilon_{\mathrm{Hg^*A}}/k = 366.8^\circ \mathrm{K.}\,,$$
$$\sigma_{\mathrm{Hg^*A}} = 3.243\ \text{Å}\,, \tag{9}$$
$$(r_{\mathrm{Hg^*A}})_{\mathrm{m}} = 3.632\ \text{Å}\,.$$

The calculated Lennard-Jones potentials for the ground state and for the excited state were plotted in Fig. 1 and compared with potentials suggested by H. Kuhn [12].

The author wishes to express his thanks to Professor A. Jabłoński for advice and interest shown during the investigations.

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БЮЛЛЕТЕНЬ
ПОЛЬСКОЙ АКАДЕМИИ НАУК
СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ И ФИЗИЧЕСКИХ НАУК

Резюме статей

ТОМ VI 1958 ВЫПУСК 2

А. ГУЛЯНИЦКИЙ, О КАРДИНАЛЬНЫХ ЧИСЛАХ СВЯЗАННЫХ С ЛОКАЛЬНО КОМПАКТНЫМИ ГРУППАМИ . . . . стр. 67—70

Локально компактная группа называется $\aleph$-компактной тогда, если она является суммой $\aleph$ своих компактных подмножеств.

В работе доказываются теоремы, что для локально компактной группы G, имеет место неравенство $\bar{G} \geqslant 2^{\theta(G)}$, где $\theta(G)$ является минимальной в отношении мощности базой открытых множеств в точке.

Отсюда для компактных групп или в общем говоря $2^{\theta(G)}$-компактных имеем $\bar{G} = 2^{\theta(G)}$.

А. ГУЛЯНИЦКИЙ, АЛГЕБРАИЧЕСКАЯ СТРУКТУРА АБЕЛЕВЫХ КОМПАКТНЫХ ГРУПП . . . . стр. 71—73

В своей книге Капланский [3] ставит проблему алгебраической характеризации тех абелевых групп, которые допускают компактную топологию. Капланский отмечает, что эта проблема подразделяется на две проблемы:

1. характеризации полных абелевых групп, допускающих компактную топологию;

2. характеризации редуцированных абелевых групп, допускающих компактную топологию.

Решение проблемы 1 дается автором в работе [2]. В настоящей заметке дается решение проблемы 2. В работе доказывается, что класс редуцированных абелевых групп, допускающих компактную топологию, является идентичным с классом всех продуктов конечных циклических групп и групп целых p-адических чисел.

В. ОКТАБА, О ЛИНЕЙНОЙ ГИПОТЕЗЕ В ТЕОРИИ НОРМАЛЬНОЙ РЕГРЕССИИ . . . . стр.75—78

На параметры β нескоррелированных случайных переменных $y_i - N\left(\sum_{j=1}^{p} x_{ij}\beta_j, \sigma\right)$ с матрицей X ранга p $(i=1,2,...,n;\ j=1,2,...,p;\ p<n)$ наложено некоторое число рестрикций линейно независимых. Для проверки линейных гипотез, приписывающих линейным комбинациям параметров неко-

торые определенные значения, служат выражения, являющиеся случайными переменными F, форма которых требует проведения минимализации для каждого определенного типа модели множественной регрессии.

Для разных типов таких моделей автор представляет в матричной форме выражения для случайных переменных F, в которых минимализация отсутствует (сравн. теоремы 1, 2 и 3).

В частности, таким путем, автор получает выражение для случайной переменной F в случае неортогональности линейных функций при единичной группировке (сравн. формулу (13)).

В настоящем докладе автор построил матрицу фундаментальных решений параболической системы уравнений в виде (1) французского текста, принимая более общие предположения, что коэффициенты уравнений (1) выполняют условия Гельдера (3), (3′), (3″). Автор сначала построил матрицу квази-решений $\{W_{\alpha,\beta}^{z,\zeta}(X, t; Y, \tau)\}$, определенную формулами (7), где функции v_α^β являются решениями системы обыкновенных дифференциальных уравнений (8). Затем, автор доказал пять теорем. Теорема 1 касается предельного свойства (10) интегралов (9), аналогичных с интегралом Пуассона-Вейерштрасса. Теорема 2 говорит о существовании производных (12) порядка $m < M$ интегралов (11), составляющих квази-потенциал объемного заряда. Теорема 3 говорит о существовании производных (14) порядка M этого потенциала. Теорема 4 касается удовлетворения уравнений (16) компонентами (11) квази-потенциала. Теорема 5 дает формулы (17) для матрицы $\{\Gamma_{\alpha\beta}\}$ фундаментальных решений системы (1). В этих формулах функции $\Phi_{\gamma,\beta}$ являются решениями системы интегральных уравнений (18), обладающих свойствами (19), (19′), (20).

В докладе автор представил в двух теоремах свойства сингулярного интеграла вида (1) французского текста. L—обозначает совокупность разомкнутых дуг на плоскости комплексной переменной, интеграл имеет смысл главного значения Коши. Функция $f(t, \tau)$, определенная для $t \in L$, $\tau \in L$ удовлетворяет неравенствам (2), (3), (4); дуги L — неравенству (5). Теорема 1 говорит, что интеграл (1) выполняет неравенство (6) в любой точке t внутри дуг L. Теорема 2 говорит, что интеграл (1) удовлетворяет условию Гельдера в виде (7), предполагая, что точка t_1 следует за точкой t в положительном неправлении дуги $a_\gamma b_\gamma$, если эти точки лежат на той же самой дуге.

Затем, автор применил эти свойства интеграла (1), доказывая в Теореме 3 существование решения интегрального уравнения (8) для достаточно малых значений параметра $|\lambda|$. Автор предположил, что заданная функция $F(t, \tau, u)$ выполняет условия (10), (11), (12) и использовал метод неподвижной точки Шаудера.

С. МРУВКА. ОБОБЩЕНИЕ ОДНОЙ ТЕОРЕМЫ, КАСАЮЩЕЙСЯ МОЩНОСТИ СОВЕРШЕННОГО КОМПАКТНОГО МЕТРИЧЕСКОГО ПРОСТРАНСТВА . . . . стр. 89—93

Хорошо известна следующая теорема:

Если X — *совершенное компактное метрическое пространство, тогда* $\overline{\overline{X}} = 2^{\aleph_0}$.

При помощи понятия характера точки (характер точки p это, по определению, наименьшее кардинальное число $\mathfrak{m}$, для которого у точки p имеется база окрестностей мощности $\mathfrak{m}$), эта теорема может быть формулирована следующим образом:

Если всякая точка компактного метрического пространства X *имеет характер* $\aleph_0$, *тогда* $\overline{\overline{X}} = 2^{\aleph_0}$.

В работе доказаны некоторые обобщения этой теоремы, а именно:

1. *Если всякая точка компактного пространства* X *имеет характер* $\geqslant \mathfrak{m}$, *тогда* $\overline{\overline{X}} \geqslant 2^{\mathfrak{m}}$.

2. *Если* X *является компактным* $\mathfrak{m}$-*почти-метрическим пространством* [3], *и всякая точка* X *имеет характер* $\mathfrak{m}$, *тогда* $\overline{\overline{X}} = 2^{\mathfrak{m}}$.

3. *Если всякая точка локально компактного пространства* X *имеет характер* $\geqslant \mathfrak{m}$, *тогда* $X \geqslant 2^{\mathfrak{m}}$.

Следствиями этих теорем являются следующие утверждения, касающиеся топологических групп:

a) *если* G — *локально компактная топологическая группа и единица группы* G *имеет характер* $\geqslant \mathfrak{m}$, *тогда* $G \geqslant 2^{\mathfrak{m}}$;

b) *если* G — *локально компактная топологическая группа и* $\mathfrak{m}$ — *наименьшее кардинальное число, для которого группа* G *является* $\mathfrak{m}$-*почти метризуемой, тогда* $G \geqslant 2^{\mathfrak{m}}$;

c) *если* G — *компактная топологическая группа и единица группы* G *имеет характер* $\mathfrak{m}$, *тогда* $\overline{\overline{G}} = 2^{\mathfrak{m}}$.

С. МРУВКА. НЕКОТОРОЕ СВОЙСТВО ГУИТОВСКОГО РАСШИРЕНИЯ νX ТОПОЛОГИЧЕСКИХ ПРОСТРАНСТВ . . . . стр. 95—96

Известно, что геховское расширение βX топологических пространств обладает следующим свойством (Y^X обозначает множество всех непрерывных функций, определенных на X, со значениями в Y):

Если Y — *компактное пространство и* $f \in Y^X$, *тогда* f *допускает расширение* $f^* \in Y^{\beta X}$.

В настоящей работе доказано аналогичное свойство гуитовского расширения νX [1]:

Если Y — Q-*пространство и* $f \in Y^X$, *тогда* f *допускает расширение* $f^* \in Y^{\nu X}$.

Намечается также возможность обобщения этих теорем.

Ч. БЕССАГА, ЗАМЕЧАНИЕ О КОНЕЧНОМЕРНЫХ УНИВЕРСАЛЬНЫХ БАНАХОВЫХ ПРОСТРАНСТВАХ . . . . стр. 97—101

Пусть n — натуральное число. Нет n-мерного банахового пространства универсального (относительно изометрии) для всех таких двумерных банаховых пространств, которых сферы являются $(2n+2)$-угольниками.

Т. ТРАЧИК, НЕКОТОРАЯ ТЕОРЕМА ОБ ОБОЗНАЧЕНИИ ГОМОМОРФИЗМОВ ПРИ ПОМОЩИ ФУНКЦИЙ . . . . стр. 102—106

В работе рассматривается произвольная σ-булевая алгебра $\boldsymbol{A}$, пространство Стона $\mathfrak{S}$ и изоморфизм Стона $\mathfrak{s}$, отображающий алгебру $\boldsymbol{A}$ на поле $\boldsymbol{S}$ подмножеств пространства Стона. Рассматривается σ-идеал $\boldsymbol{N}$ подмножеств пространства $\mathfrak{S}$ генерированный множествами вида $\mathfrak{s}(A)-\sum_{n=1}^{\infty}\mathfrak{s}(A_n)$, где $A, A_n \epsilon \boldsymbol{A}$ $A=\sum_{n=1}^{\infty}A_n$, а также рассматривается σ — поле $\boldsymbol{Z}$ подмножеств из пространства $\mathfrak{S}$ вида $Z=S-N_1+N_2$, где $S \epsilon \mathfrak{S}$, $N_1, N_2 \epsilon \boldsymbol{N}$.

Главным результатом рассуждений является следующая теорема: *Если $\boldsymbol{X}/\boldsymbol{I}$ обозначает произвольную σ-фактор алгебру, то всякий σ-гомоморфизм h, отображающий $\boldsymbol{Z}/\boldsymbol{N}$ в $\boldsymbol{X}/\boldsymbol{I}$, порождается функцией тогда и только тогда, когда существует гомоморфизм n, отображающий $\boldsymbol{X}/\boldsymbol{I}$ в $\boldsymbol{X}$, такой, что $n([X]) \epsilon [X]$, где $X \epsilon \boldsymbol{X}$.*

Кроме того, дается пример σ-фактор алгебры $\boldsymbol{X}/\boldsymbol{I}$, для которой условие, данное в главной теореме, не выполняется.

М. ЧАЙКОВСКИЙ и Я. ГЖИВАЧ, ЗАВИСИМОСТЬ СТЕПЕНИ ПОЛЯРИЗАЦИИ ФЛУОРЕСЦЕНЦИИ ОТ КОНЦЕНТРАЦИИ ЛЮМИНЕСЦЕНТНЫХ МОЛЕКУЛ В ПЛЕКСИГЛАСС-ФОСФОРАХ . . . . . стр. 107—111

Приготовлены жесткие растворы нафтацена и эозина в метакриловом метиле с целью исследования зависимости степени поляризации флуоресценции от концентрации красителя.

Обнаружен рост степени поляризации одновременно с ростом концентрации красителя.

Предположена теоретическая интерпретация наблюденной зависимости поляризации флуоресценции от концентрации красителя.

Р. БАУЭР и А. БОНЧИНСКИЙ, ЗАВИСИМОСТЬ СООТНОШЕНИЯ ВЫХОДА ФОСФОРЕСЦЕНЦИИ К ВЫХОДУ ФЛУОРЕСЦЕНЦИИ ОТ ДЛИНЫ ВОЛНЫ ВОЗБУЖДАЮЩЕГО СВЕТА . . . . стр. 113—117

Исследована зависимость соотношения выхода фосфоресценции к выходу флуоресценции от длины волны возбуждающего света.

Констатировано, что вероятность перехода $F \to M$ зависит от длины волны возбуждающего света. Это явление объясняется при помощи потенциальных кривых Франка-Кондона.

В приборе построенном К. Антоновичем, согласно проекту А. Яблонского, введены некоторые улучшения, а затем при помощи этого прибора интегрировано уравнение Шредингера с потенциалом

$$V_{(r)} = \frac{c_2}{r^8} - \frac{c_1}{r^6}$$

для случая взаимодействия атома ртути с аргоном. Полученные функции для дискретного спектра были нормированы, а затем была проверена их ортогональность. Введенные изменения позволили избежать необходимости (сугеррированной в предшествующих работах К. Антоновича [1], [2], [3]) мечения прибора путем решения известной проблемы.

Вычислены параметры потенциала Леннард-Джонса (6—12) для пары атомов Hg и A. Определены постоянные на основании, находящихся в литературе, экспериментальных данных для двух случаев:

1. когда атом ртути находится в нормальном состоянии $^1S_0$
2. когда атом ртути находится в возбужденном состоянии $^3P_1$. Вычисленные потенциалы представлены на графике и сопоставлены с кривыми Куна.

TABLE DES MATIÈRES

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